Mathematics - II

Differential equation

Subject Code : 101204

Credit : 4

Lecture : 3 : 1 : 0

Books :- ① Manish Goyal & N.P. Bali
② Erwin Kreyszing : Advance engg. Maths

Differential equation :-

An equation involving derivative co-efficient and independent variable, dependent variable is known as differential equation.

The differential eqn classify in two parts.
(i) ordinary diff^l eqn.
(ii) partial diff^l eqn.

① Ordinary diff^l eqn :-
when the dependent variable depends on a single independent variable then we says that diff^l is ordinary diff^l eqn. ex.

$$\frac{dy}{dx} + y = 0$$

Partial diff^l eqn:-

when the dependent variable depends are two or more independent variable then we say that diff^l is partial diff^l eqn.

ex.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 4$$

Order of a diff^l eqn:-

The order of the diff^l eqn is the order of a highest derivative present in diff^l eqn.

Degree of a diff^l eqn:-

The degree of a diff^l eqn is the degree (power) of the highest order derivative

① The order of the diff^l eqn

$$30\frac{d^2y}{dt^2} + 2\left(1-\left(\frac{dy}{dt}\right)^3\right)^{1/2} = 0 \text{ is}$$

(i) 0 (ii) 1 (iii) 2 (iv) 3

The order and degree of diff^l eqn

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^{3/2}$$

order = 2, degree = 12

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Exact diff^l eqn:-

Any 1st order diff^l eqn can be written as

$M(x,y)dx + N(x,y)dy = 0$

or $Mdx + Ndy = 0$

$\left[\because P\frac{dy}{dx} + Qy = R\right.$

$Pdy + Qy\,dx = R\,dx$

$Pdy + (Qy - R)dx = 0$

$Ndy + Mdx = 0$

Then the given diff^l eqn will be exact if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

for ex

$$\frac{1+e^{x/y}}{e^{x/y}\left(\frac{x}{y}-1\right)} = \frac{dy}{dx}$$

$\Rightarrow (1+e^{x/y})dx = e^{x/y}\left(\frac{x}{y}-1\right)dy$

$\Rightarrow (1+e^{x/y})dx - e^{x/y}\left(\frac{x}{y}-1\right)dy = 0$

$\Rightarrow (1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

which is $Mdx + Ndy = 0$ type

Then $M = 1+e^{x/y}$

$N = e^{x/y}\left(1-\frac{x}{y}\right)$

For exactness

$$\boxed{\frac{\partial M}{\partial y} = e^{x/y} \times \left(-\frac{1}{y^2}\right)\cdot x}$$

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$$\frac{\partial N}{\partial x} = e^{x/y} \times \left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right) \cdot e^{x/y}\left(\frac{1}{y}\right)$$

$$= -\frac{e^{x/y}}{y} + \frac{e^{x/y}}{y} - \frac{x}{y^2} e^{x/y}$$

$$\boxed{\frac{\partial N}{\partial x} = -\frac{x}{y^2} e^{x/y}}$$

$\therefore$ Given eqn. is exact.

Solution of the exact diff^{l} eqn.

For give exact diff^{l} eqn we get the solution as follows.

$$\boxed{\underbrace{\int M\,dx}_{\text{treating } y \text{ as a const}} + \underbrace{\int N\,dy}_{\text{term free from } x} = 0 = c}$$

For ex. Soln. of the diff^{l} eqn $\frac{1+e^{x/y}}{e^{x/y}\left(\frac{x}{y}-1\right)} = \frac{dy}{dx}$ will be

$$\int M\,dx + \int N\,dy = c$$

$$\int (1+e^{x/y})\,dx + \int 0 = c$$

$$\boxed{x + \frac{e^{x/y}}{1/y} = c}$$

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ex $(y\sin(xy) + xy^2\cos(xy))dx + (x\sin(xy) + x^2y\cos(xy))dy = 0$

soln: ~~sol~~ Given eqn is $Mdx + Ndy = 0$ type
Hence

$$M = y\sin xy + xy^2\cos xy$$

$$N = x\sin xy + x^2y\cos xy$$

For exactness

$$\frac{\partial M}{\partial y} = y\cos(xy)\cdot(x) + \sin xy + xy^2(-\sin(xy))x + \cos(xy)\cdot(2xy)$$

$$= xy\cos(xy) + \sin xy - x^2y^2\sin(xy) + 2xy\cos(xy)$$

$$\frac{\partial N}{\partial x} = x\cos(xy)\,y + \sin(xy) + x^2y(-\sin(xy))\,y + \cos(xy)\,2xy$$

$$= xy\cos(xy) + \sin(xy) - x^2y^2\sin(xy) + 2xy\cos xy$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence given diff eqn is exact

Soln. $\int M\,dx + \int N\,dy = c$

$$\int\{y\sin(xy) + xy^2\cos(xy)\}\,dx = c$$

$$y\int\sin(xy)\,dx + y^2\int x\cos(xy)\,dx = c$$

$$y\left(-\frac{\cos(xy)}{y}\right) + y^2\left[x\int\cos(xy)\,dx - \int\left[1\cdot\int\cos(xy)\,dx\right]\right] =$$

① Solve. $\{2xy\cos x^2 - 2xy + 1\}dx + \{\sin x^2 - x^2 + 3\}dy = 0$

② Solve $\{1 + \log(xy)\}dx + \{1 + \frac{x}{y}\}dy = 0$

Solⁿ ① Here

$$M = 2xy\cos x^2 - 2xy + 1$$
$$N = \sin x^2 - x^2 + 3$$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$$

$$\frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence give eqn will be exact

∴ sol^n will be

$$\int M dx + \int N dy = c$$

$$\int (2xy\cos x^2 - 2xy + 1)dx + \int 3dy = c$$

$$= 2y\int x\cos x^2 dx - 2y\int x dx + \int dx + 3\int dy = c$$

Let $x^2 = t$
$2x dx = dt$

$$y\int \cos t\, dt - 2y\int x dx + \int dx + 3\int dy = c$$

$$y\sin t - \frac{2yx^2}{2} + x + 3y = c$$

$$\boxed{y\sin x^2 - yx^2 + x + 3y = c}$$

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Sol^n (2)

given $\{1+\log(xy)\}dx+\{1+\frac{x}{y}\}dy=0$

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$Mdx+Ndy=0$

$M=1+\log(xy)=1+\log x+\log y$

$N=\left(1+\frac{x}{y}\right)$

$\frac{\partial M}{\partial y}=\frac{1}{y}$

$\frac{\partial N}{\partial x}=\frac{1}{y}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Hen the given eqn will be exact.

$\therefore$ Sol^n.

$\int M dx+\int N dy=c$

$\int(1+\log x+\log y)dx+\int 1 dy=c$

$\int dx+\int\log x\,dx+\log y\int dx+\int dy=c$

$x+\{\log x\cdot x-\int\frac{1}{x}x\,dx\}+(\log y)x+y=c$

$=x+x\log x-\cancel{x}+x\log y+y=c$

$x(\log x+\log y)+y=c$

$x\log(xy)+y=c$

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7) Integrating factor:
An diff^{l} eqn which is not exact, can be made exact by multiplying a function $f(x,y)$ is known as integrating factor.

Rule - I If $\frac{1}{N}\left\{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right\}$ is funtion of x only then integrating factor will be

$$e^{\int F(x)dx}$$

ex. Solve $(2x\log x - xy)dy + 2y\,dx = 0$

sol^{n}. Here $N = 2x\log x - xy$

$M = 2y$

$$\frac{\partial M}{\partial y} = 2$$

$$\frac{\partial N}{\partial x} = \frac{2x}{x} + \log x \, 2 - y$$

$$= 2(1 + \log x) - y$$

$$\therefore \frac{1}{N}\left\{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right\}$$

$$\frac{1}{2x\log x - xy}\left\{\cancel{2} - \cancel{2} - 2\log x + y\right\}$$

$$-x\frac{1}{\cancel{(-\log x + y)}}\cancel{(-2\log x + y)}$$

$$= -\frac{1}{x}$$, which is function x only.

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I.F. will be

$e^{\int F(x)dx}$

$e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$

$= \frac{1}{x}$

$\therefore$ multiplying ~~factor~~ I.F. $\frac{1}{x}$ in given diff^{l} eqn

$(2\log x - y)dy + 2\frac{y}{x}dx = 0$

check $\left\{ \frac{\partial M'}{\partial y} = \frac{2}{x}, \frac{\partial N'}{\partial x} = \frac{2}{x} \right.$

Thus given eqn is exact.

Hence the soln. will be

$\int_{y\ \text{const}} M'dx + \int_{\text{terms not containing}} N'dy = c$

$\int 2\frac{y}{x}dx + \int -y\,dy = c$

$2y\int \frac{1}{x}dx - \int y\,dy = c$

$$\boxed{2y\log x - \frac{y^2}{2} = c}$$

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Method - II If $\frac{1}{M}\left\{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right\}$ is function of y then I.F will be $e^{\int f(y)\,dy}$

ex: Solve $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

Sol^n. $M = y^4+2y$

$N = xy^3+2y^4-4x$

$\frac{\partial M}{\partial y} = 4y^3+2$

$\frac{\partial N}{\partial x} = y^3-4$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$

$4y^3+2-y^3+4$

$3y^3+6$

$3(y^3+2)$

Now $\frac{1}{M}\left\{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right\}$

$\frac{1}{y^4+2y}\left(y^3-4-4y^3-2\right\}$

$= \frac{3(y^3+2)}{y(y^3+2)} = \frac{-3}{y}$

$\therefore IF = e^{\int F(y)\,dy}$

$e^{\int \frac{-3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}}$

$= y^{-3} = \frac{1}{y^3}$

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Hence multiply by $\frac{1}{y^3}$ in given diff^l eqn

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$$

which is exact eqn

$$\int_{y\ const} M dx + \int N dy = c$$

terms not contain x

$$\int \left(y + \frac{2}{y^2}\right) dx + \int 2y\, dy = c$$

$$\left(y + \frac{2}{y^2}\right) \int dx + 2 \int y\, dy = c$$

$$\left(y + \frac{2}{y^2}\right) x + \frac{2y^2}{2} = c$$

$$= \left(y + \frac{2}{y^2}\right) x + y^2 = c$$

‡ Rule - 3: If give diff^l eqn is of the form $y F(xy)\, dx + x\, g(xy)\, dy = 0$ then I.F. will be

$$\frac{1}{Mx - Ny}, \text{ provided } Mx - Ny \neq 0$$

ex. Solve $(xy^2 + 2x^2y^3)\, dx + (x^2y - x^3y^4)\, dy = 0$

sol^n: ~~M~~ = given diff^l eqn ~~wh~~ can be written as

$$y(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)\, dy = 0$$

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Then

$$IF = \frac{1}{Mx - Ny}$$

$$\frac{1}{(xy^2 + 2x^2y^3)x - (x^2y - x^3y^2)y}$$

$$= \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + 2x^3y^3}$$

$$\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying ~~given~~ IF in given diff^{l} eq^{n}

$$\frac{1}{3x^3y^3}\left[(xy^2 + 2x^2y^3)\,dx + (x^2y^2 - x^3y^2)\,dy\right] = 0$$

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

which is exact diff^{l}

$$\int M\,dx + \int N\,dy = c$$

y const terms not containg x

$$\int\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{y}\,dy = c$$

$$\frac{1}{y}\int\frac{1}{x^2}dx + 2\int\frac{1}{x}\,dx - \int\frac{1}{y}\,dy = c$$

$$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c$$

$$= \boxed{-\frac{1}{xy} + 2\log x - \log y = c}$$

$\frac{1}{y^2}$

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Reme: 4 10 Given diff^l eqn is of the homogeneous form $f(x,y)dx + xg(xy)dy = 0$

then I.F. will be $\frac{1}{Mx+Ny}$, provided $Mx+Ny \neq 0$

ex. solve. $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

(ii) $(5x^4+3x^2y^2-2xy^3)dx + (2x^3y-3x^2y^2-5y^4)dy = 0$

soln. (ii) given diff eqn is homogeneous

Hint $\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$

$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

soln (i) given diff^l eqn can be written as

$xy^2dy = (x^3+y^3)dx$

$\Rightarrow (x^3+y^3)dx - xy^2dy = 0$

given diff^l eqn is homo. diff^l eqn

Hence IF will be $\frac{1}{Mx+Ny}$

$\frac{1}{(x^3+y^3)x + (-xy^2)y}$

$\frac{1}{x^4+xy^3-xy^3} = \frac{1}{x^4}$

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Multiplying given diff. eq. by $\frac{1}{x^4}$

$$\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}\,dy = 0$$

which is exact diff^l eq^n

~~Hence~~ Sol^n.

$$\underset{y\ \text{const}}{\int} M\,dx + \int N\,dy = c$$

terms not contaning x

$$\int\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx + \int 0\,dy = c$$

$$\int \frac{1}{x}\,dx + y^3 \int x^{-4}\,dx$$

$$\log x + y^3 \frac{x^{-4+1}}{-4+1} = c$$

$$\log x + \frac{y^3}{3x^3} = c$$

$$= \log x - \frac{y^3}{3x^3} = c$$

Rule-5: If the diff^{l} eqn is of the form

form $x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a' y\,dx + b' x\,dy) = 0$

then I.F. will be

$$x^h . y^k$$

where h and k are related by

$$\frac{m+h+1}{a} = \frac{n+k+1}{b} \text{ and } \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

example. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Sol^n: $y^3 dx - 2x^2 y\,dx + 2xy^2 dy - x^3 dy = 0$

$$y^2(y\,dx + 2x\,dy) + x^2(-2y\,dx - x\,dy) = 0$$

Comparing with

$$x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a' y\,dx + b' x\,dy) = 0$$

Here,

$m = 0, n = 2, a = 1, b = 2$

$m' = 2, n' = 0, a' = -2, b' = -1$

then by relation

$$\frac{m+h+1}{a} = \frac{n+k+1}{b}$$

and $$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{0+h+1}{1} = \frac{2+k+1}{2} \;\&\; \frac{2+h+1}{-2} = \frac{0+k+1}{-1}$$

$\Rightarrow$ $2h + 2 = 3 + k$ and $3 + h = 2k + 2$

$\Rightarrow$ $2h - k = 1$ and $h - 2k = -1$

Now $2h - k = 1$

$h - 2k = -1 \quad \times 2$

$2h - k = 1$

$-2h + 4k = 2$

$3k = 3$

$k = 1, h = 1$

$\therefore$ I.F $= x^1 y^1 = xy$

Now multiply xy in given diff^l eq^n

$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0$

which is exact and its sol^n is

$$\underset{y \text{ as const}}{\int M dx} + \underset{\text{terms not containing } x}{\int N dy} = c$$

$$\int (xy^4 - 2x^3y^2)dx + \int 0\, dy = c$$

$$= y^4 \int x\, dx - 2y^2 \int x^3 dx = c$$

$$\frac{y^4x^2}{2} - \frac{2y^3x^4}{4} = c$$

$$x^2y^4 - x^4y^3 = 2c$$

$$\boxed{x^2y^4 - x^4y^3 = 2c} \text{ Ans}$$

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1st order linear diff^l eqn:-

A diff^l eqn of the form

$\frac{dy}{dx} + Py = Q$ is known as 1st order linear diff^l eqn. where P and Q are function of x only or constant

Solution:

Step-I: Convert the given diff^l eqn into standard linear diff^l eqn

Step-II. Find I.F $= e^{\int P dx}$

Step-III. Solution will be

$$y \times (I.F) = \int Q.(I.F) dx + C$$

ex: Solve $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

sol^n.

$$\frac{dy}{dx} - \frac{y}{(x+1)} = \frac{e^x(x+1)^2}{x+1}$$

$$P = -\frac{1}{(x+1)}, \quad Q = e^x(x+1)$$

$$I.F = e^{\int P dx}$$

$$e^{\int -\frac{1}{x+1} dx} = e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}}$$

$$\frac{1}{x+1}$$

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Sol^n. (1)

$$y \times (I.F) = \int Q.(I.F) \cdot dx + C$$

$$y \cdot \frac{1}{x+1} = \int e^x (x+1) \times \frac{1}{x+1} dx + C$$

$$\frac{y}{x+1} = \int e^x dx + C$$

$$\frac{y}{x+1} = e^x + C$$

$$y = e^x(x+1) + C(x+1)$$

$$\boxed{y = e^x(x+1) + C(x+1)}$$

Solve. $\sin x \frac{dy}{dx} + 2y = \tan^3\left(\frac{x}{2}\right)$

Sol^n:

$$\frac{dy}{dx} + \frac{2}{\sin x} y = \frac{\tan^3\left(\frac{x}{2}\right)}{\sin x}$$

which is linear diff^l eq^n

$$P = \frac{2}{\sin x}, \quad Q = \frac{\tan^3\left(\frac{x}{2}\right)}{\sin x} = \frac{\tan^2\left(\frac{x}{2}\right)}{2\sin\frac{x}{2} \cdot \cos\frac{x}{2}}$$

$$I.F = e^{\int P dx} = e^{2\int \frac{1}{\sin x} dx} = e^{2\int \operatorname{cosec} x \, dx}$$

$$= e^{2\log \tan\left(\frac{x}{2}\right)} = \tan^2\left(\frac{x}{2}\right)$$

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Soln.

$$y \cdot \tan^2\left(\frac{x}{2}\right) = \int \frac{\tan^3\left(\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}} \times \tan^2\left(\frac{x}{2}\right) dx + c$$

$$= \int \frac{\tan^4\left(\frac{x}{2}\right)}{2\sin\frac{x}{2} \cdot \cos\frac{x}{2}} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} dx + c$$

$$= \int \tan^4\left(\frac{x}{2}\right) \cdot \sec^2\left(\frac{x}{2}\right) dx + c$$

Let $\tan\frac{x}{2} = t$

$$\sec^2\frac{x}{2} \times \left(\frac{1}{2}\right) dx = dt$$

$$= \frac{1}{2}\int t^4 \cdot (2\,dt) + c$$

$$\frac{t^5}{5} + c$$

$$y \tan^2\left(\frac{x}{2}\right) = \frac{\tan^5\left(\frac{x}{2}\right)}{5} + c$$

Q/ve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Soln. $\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

$P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

$I.F = e^{\int P\,dy}$

$e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}(y)}$

Sol^n. $x \cdot e^{\tan^{-1}(y)} = \int \frac{\tan^{-1}y}{1+y^2} \times e^{\tan^{-1}y}\,dy + C$

let $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$dy = (1+y^2)\,dt$

$\int \underset{I}{t} \cdot \underset{II}{e^t}\,dt + c$

$t\int e^t\,du - \int e^t\,dt + c$

$t \cdot e^t - e^t + c$

$x \times e^{\tan^{-1}(y)} = \tan^{-1}(y)\,e^{\tan^{-1}(y)} - e^{\tan^{-1}(y)} + c$

$x e^{\tan^{-1}(y)} = \tan^{-1}(y)\,e^{\tan^{-1}(y)} - e^{\tan^{-1}(y)} + c$

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Bernoulle equation:
A diff^{l} eqn the form

$$\frac{dy}{dx} + Py = Qy^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \quad \text{--- (1)}$$

let $\frac{1}{y^{n-1}} = z \Rightarrow y^{1-n} = z$

diff w.r.t x

$$(1-n)y^{1-n-1}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1-n}{y^n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + zP = Q$$

$$\frac{dz}{dx} + (1-n)P.z = (1-n)Q$$

which is is linear eqn

Hence we can find the soln

(iv) Clairauts form

A diff^l eqn of form $y = xp + f(p)$ is known as clairauts form where $f(p)$ is function of p then its solution will be found by replacing $p = c$

eg solve $p = \log(px - y)$

soln. given eqn can be written as

$$px - y = e^p$$

$$y = px - e^p$$

which is clairauts form

Hence its soln. will be $y = cx - e^c$

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Second order diff^l eqn with ~~variable~~ constant coefficient:

A diff^l eqn which is form

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = a_3$$ is called

2nd order diff^l eqn with constant coefficient, where a_1, a_2, and a_3 are constant

ex ① $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = 0$

② $\frac{d^2y}{dx^2} - y = 0$

③ $y'' + 2y' = 4$

ex. Solve the diff^l eqn $y'' - 3y' + 2y = 0$

Sol^n Let $Dy = \frac{dy}{dx}$, & $D^2y = \frac{d^2y}{dx}$

Then given eqn can be written as

$$D^2y - 3Dy + 2y = 0$$

$$(D^2 - 3D + 2)y = 0$$

Auxilliary Equation (A.E)

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m + m + 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2, -1$$

Sol^n:

$$y = c_1 e^{2x} + c_2 e^{x}$$

General formula for complimentary function(C.F)

① If roots are different real & i.e $m_1 \neq m_2 \neq m_3 \neq$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots$$

② If some real roots are equal i.e. $m_1 = m_2 = m_3 \neq m_4 \neq m_5$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x}$$

③ If the roots are complex number ~~then~~ ~~C.F. will~~ be i.e $\alpha \pm i\beta$ then C.F will be

$$y = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$$

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H) Short methods to find particular function (PI)

① Let the diff^{l} eqn be

$$a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=R(x)$$

(a) If $R=e^{ax}$
then particular integral will be
$\frac{1}{F(D)}e^{ax}=\frac{1}{F(a)}e^{ax}$, provided $F(a)\neq 0$

(b) If $F(a)=0$, then
P.I will be $\frac{1}{F(D)}e^{ax}=\frac{x^n}{n!}\cancel{e^{ax}}$

$=\frac{x^n}{n!}e^{ax}$

ex. Solve $y^{iv}+5y'''+6y''-4y'-8y=e^{-2x}+2e^{-x}+3e^{x}-3$

Soln,
$(D^4+5D^3+6D^2-4D-8)y=e^{-2x}+2e^{-x}+3e^{x}-3$

A.E
$m^4+5m^3+6m^2-4m-8=0$
By hit and trial method

$m=1$
$m=-2$ $(m+2)=0$
$m^4+2m^3+3m^3+6m^2-4m-8=0$
~~$m^3(m+2)+3m^2(m+2)-4(m+2)=0$~~

Solution of second order diff^{l} eqn with variable Coefficient:

Method (1) when one solution is known
let the second order diff^{l} eqn with variable coefficient is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad -①$$

let u is one soln given then couple soln. ~~u~~ will be

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\cdot\frac{du}{dx} + v\frac{d^2u}{dx^2}$$

Put the value of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in eqn①

$$\left(u\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx}\right) + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Qu.v = R$$

$$v\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+u\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}\right)+2\frac{du}{dx}\frac{dv}{dx}=R$$

$$0+u\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}\right)+2\frac{du}{dx}\frac{dv}{dx}=R \quad \left[\because u \text{ is a sol}^n\right.$$

$$\frac{d^2v}{dx^2}+P\frac{dv}{dx}+\frac{2}{u}\frac{du}{dx}\frac{dv}{dx}=\frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

Let $\frac{dv}{dx}=z$, $\frac{d^2v}{dx^2}=\frac{dz}{dx}$

then $\frac{dz}{dx}+\left(P+\frac{2}{u}\frac{du}{dx}\right)z=\frac{R}{u}$

which is linear di eqn

Rule To find the one solution (u) of the diff^l eqn.

Let the given diff^l eqn is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

(i) $1+P+Q=0$ then $u=e^x$

(ii) $1-P+Q=0$ then $u=e^{-x}$

(iii) $a^2+aP+Q=0$ then $u=e^{ax}$

where a is any constant

(iv) $P + Qx = 0$ then $u = x$

(v) $m(m-1) + Pmx + Qx^2 = 0$ then $u = x^m$

(vi) $2 + 2xP + Qx^2 = 0$ then $u = x^2$

Solve. $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is an integral in the complementary function

: Let $u = e^{x^2}$ then for other soln v, when we put $y = uv$ then we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-4x + \frac{2}{\cancel{e^{x^2}}} \cdot 2x \times \cancel{e^{x^2}}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} = 0$$

$$\frac{dv}{dx} = c_1.$$

$$v = c_1x + c_2$$

$\therefore$ $y = uv$

$$\boxed{y = e^{x^2}(c_1x + c_2)}$$

Method of solving linear diffˡ eqⁿ with variable coefficient by changing the indepent variable;

Rules - I

Step-I convert the give diffˡ eqⁿ in to

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Step-II Find R_1, Q_1 and P_1 by

$$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

And ${}^{O.D.E}_{CASE}$ eqn $\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$

Step-III

Method-I

solve for $P_1=0$

Method-II

Solve for $Q_1=$ constant in terms of z
then the solve the diff^l eqn

$$\frac{d^2y}{dz^2}+P_1\frac{dz}{dz}+Q_1y=R_1$$

ex $\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y\,cosec^2x=0$

Given that

$$\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y\,cosec^2x=0$$

$P=\cot x,\quad Q=4\,cosec^2x\ ,R=0$

$$P_1=\frac{\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right]}{\left(\frac{dz}{dx}\right)^2}=0\quad(\text{say})$$

$$\left[\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right]=0$$

$$\left[\frac{d^2z}{dx^2}+\cot x\frac{dz}{dx}\right]=0$$

Let $\frac{dz}{dx}=t$

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Second order diff^{l} eqn with variation of parameter: let diff^{l} eqn be $y''+Py'+Qy=R$

Step-I Find the complementary function C.F as Ay_1+By_2, where A and B are constant.

Step-II Let its particular integral be uy_1+vy_2 where ~~u and v~~

$$u=\int\frac{-Ry_2}{W}dx \quad , \quad v=\int\frac{Ry_1}{W}dx$$

where W is called Wronskian

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$$

Substitute the value of u & v in uy_1+vy_2

Step-III The complete solution will be

$$y=C.F+P.I$$

ex Solve $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x$

Sol^{n} Given diff^{l} eqn is const coefficient

$A.E = m^2+1=0$

$m=\pm i$

$C.F = A\cos x+B\sin x$

then let $y_1=\cos x$, $y_2=\sin x$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin x$$
$$= 1$$

$$u = \int \frac{-y_2 R}{W} dx = \int \frac{-\sin x \cdot \operatorname{cosec} x}{1} dx = \int -dx = -x$$

$$v = \int \frac{y_1 R}{W} dx = \int \frac{\cos x \operatorname{cosec} x}{1} dx$$

$$= \int \cot x \, dx = \log \sin x$$

$$P.I = u y_1 + v y_2 = -x \cos x + \log \sin x \sin x$$

Sol^n.

$$y = C.F. + P.I$$

$$\boxed{A \cos x + B \sin x - x \cos x + \log \sin x \cdot \sin x}$$

Solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$

Q^r $A.E = m^2 - 3m + 2 = 0$

$$m^2 - 2m - m + 2 = 0$$
$$m(m-2) - 1(m-2) = 0$$
$$m = 1, 2$$

$$C.F = c_1 e^{2x} + c_2 e^{x}$$

Cauchy's - Euler Homogeneous Linear diff. eqn

The diff eqn of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = \phi(x)$$

where $a_0, a_1, \ldots a_n$ are constant and $\phi(x)$ is any function of x

For ex. $x^3 \frac{d^3 y}{dx^3} - 2x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 7y = x^2$

Step.

I Let $x = e^z \Rightarrow z = \log x$

(ii) $x^3 y''' = D(D-1)(D-2)$

$x^2 y'' = D(D-1)$

$x y' = D$

(iii) Find A.E

(iv) Find C.F

(v) Find P.I

(vi) replace z by $\log x$

where $\frac{d}{dz} = D$

Power Series Method:

Let the second order linear diff^{l} eqn be

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

where P_0, P_1 and P_2 are polynomial in x

If $P_0 \neq 0$ for any ~~non-zero~~ some point ~~the~~ x_0 then this point is called ordinary point

Singular point:

Let the second order linear diff^{l} eqn be

$$\frac{d^2y}{dx^2} + Q_1 \frac{dy}{dx} + Q_2 y = 0$$

a point $x=a$ is said to singular point if Q_1 or Q_2 is ~~not~~ not analytic ($Q_1 = \infty$ or $Q_2 = \infty$)

Singular points are two ~~point~~ types.

(i) Regular singular points

(ii) Irregular singular points.

Regular Singular points

Let $x=a$ be a singular point.

Let $Q_1' := (x-a) Q_1$ and $Q_2' := (x-a)^2 Q_2$

then $x=a$ is said to be regular singular point

if Q_1' and Q_2' is analytic

ie $Q_1' \neq \infty$ and $Q_2' \neq 0$

→ Irregular Singular point:

$x=a$ is said to be irregular singular point if either $Q_1' = \infty$ or $Q_2' = \infty$

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Power Series Solution at $x = 0$ (Ordinary Point)

Working rule

(1) Let soln. the $y = a_0 + a_1 x + a_2 x^2 + \dots +$

$$= \sum_{n=0}^{\infty} a_n x^n \quad (a_0, a_1 \text{ are const to be determined})$$

(2) Find $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

(3). Substitute these expression into given diff eqn

(4) Comparing the diff. coefficient of x find the value of $a_0, a_1, a_2 \dots$

(5) put the values of $a_0, a_1, a_2 \dots$ n

$$y = \sum_{n=0}^{\infty} a_n x^n$$

x Solve the soln. of the diff. eqn

$\frac{d^2y}{dx^2} + x^2 y = 0$ in power series.

Soln.: Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 =$

$$= \sum_{n=0}^{\infty} a_n x^n \quad (1)$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 = \dots = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

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Legendre diff^{l} eqn:

The diff^{l} eqn $(1-x^2)y''-2xy'+n(n+1)y=0$ ①
is known as Legendre diff^{l} eqn.
where n is any constant.

The above eqn can be written as

$$\frac{d}{dx}\left[(1-x^2)y'\right]+n(n+1)y=0$$

Solution (Frobeneous method)

Let the solution in series of descending order be

$$y=\sum_{r=0}^{\infty} a_r x^{m-r}$$

$$y'=\sum_{r=0}^{\infty} a_r(m-r)x^{m-r-1}$$

$$y''=\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}$$

Substitute these values in eqn ①

$$(1-x^2)\left\{\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}-\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r}\right.$$

$$-2\sum a_r(m-r)x^{m-r}+n(n+1)\sum a_r x^{m-r}=0$$

$$\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}-\sum_{r=0}^{\infty} a_r\{(m-r)(m-r-1)+2(m-r)-n(n+1)\}x^{m-r}=0$$

$$\text{''} \quad -\sum_{r=0}^{\infty} a_r\{(m-r)(m-r+1)-n(n+1)\}x^{m-r}=0$$

Rodrigues's formula:

$$\boxed{P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2-1)^n \quad \text{--- (1)}}$$

proof: Let

$$v = (x^2-1)^n$$

$$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x) = 2nx(x^2-1)^{n-1}$$

multiply by (x^2-1) on both sides

$$(x^2-1)\frac{dv}{dx} = 2nx(x^2-1)^n$$

$$(x^2-1)\frac{dv}{dx} = 2nxv \quad \text{--- (2)}$$

also we know that Leibnitz's Rule.

$$\frac{d^n[uv]}{dx^n} = uv_n + {}^nC_1 u' v^{n-1} + {}^nC_2 u'' v^{n-2} + \cdots + u_n v$$

then diff. eqn (2) $(n+1)$ times

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + {}^{n+1}C_1(2x)\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}C_2(2)\frac{d^n v}{dx^n} + 0$$

$$= 2n\left\{(x)\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}C_1 (1)\frac{d^n v}{dx^n}\right\}$$

$$= (x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + (n+1)2x\frac{d^{n+1}}{dx^{n+1}} + \frac{(n+1)n(2)}{2}\frac{d^n v}{dx^n} = 2nx\frac{d^{n+1}v}{dx^n}$$
$$+ 2n(n+1)\frac{d^n v}{dx^n}$$

$$= (x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + \frac{d^{n+1}v}{dx^{n+1}}\left(2nx + 2x - 2nx\right) + \frac{d^n v}{dx^n}\left[n(n+1) - 2n(n+1)\right] = 0$$

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{n+1}v}{dx^{n+1}} - n(n+1)\frac{d^n v}{dx^n} = 0$$

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} = 0.$$

Let $\frac{d^n v}{dx^n} = y$, then

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

that is $y = \frac{d^n v}{dx^n}$ is one solution

$$\therefore P_n(x) = cy = c\frac{d^n v}{dx^n} \quad \text{—— (11)}$$

where c is a constant to be determi[ned]

also $v = (x^2-1)^n = (x+1)^n(x-1)^n$ —— 4

diff^d eqn (4) n-times

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Orthogonal properties of Legendre Function $P_n(x)$

$$\int_{-1}^{1} P_n P_m \, dx = \begin{cases} 0 & , m \neq n \\ \frac{2}{2n+1} & , m = n \end{cases}$$

proof:

Let $P_n(x)$ is a solution of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

and $(1-x^2)z'' - 2xz' + m(m+1)z = 0$ --- (2)

multiply in eqn (1) by z and in eqn (2) by y and subtract then

$$(1-x^2)\{zy'' - yz''\} - 2x\{zy' - yz'\} + \{n(n+1) - m(m+1)\}zy = 0$$

$$(1-x^2)\{(zy'' + z'y' - z'y' - yz'')\} - 2x\{zy' - yz'\} + \{n(n+1) - m(m+1)\}yz = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)(zy' - yz')\right] + \{n^2 + n - m^2 - m\}yz = 0$$

$$(1-x^2)\left[zy'' + z'y' - yz'' - y'z'\right]$$

integrating from -1 to 1 above eqn

$$\left[(1-x^2)(zy' - yz')\right]_{-1}^{1} + (n-m)(n+m+1)\int_{-1}^{1} yz \, dx = 0$$

$$0 + (n-m)(n+m+1)\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0$$

$$\int_{-1}^{1} P_n(x) \, P_m(x) = 0$$

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Part second when $m = n$, then we have to prove.

$$\int_{-1}^{1} P_n^2 \, dx = \frac{2}{2n+1}$$

we know that

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{--- (a)}$$

Squaring both sides

$$(1 - 2xz + z^2)^{-1} = \sum_{n=0}^{\infty} z^{2n} P_n^2 + 2z^{n+m} \cdot P_n(x) P_m(x)$$

Integrating both sides

$$\int_{-1}^{1} \frac{1}{1 - 2xz + z^2} dx = \int_{-1}^{1} \sum z^{2n} P_n^2(x) dx + \int_{-1}^{1} 2z^{m+n} P_n(x) P_m(x) dx$$

$$\Rightarrow \frac{-1}{2z} \left[\log (1 - 2xz + z^2) \right]_{1}^{-1} = \sum z^{2n} \int_{-1}^{1} P_n^2(x) dx$$

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Recurrence Formulae.

① $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}, \quad n \geq 2$

Or $\quad xP_n = \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1}$

② $nP_n = xP'_n - P'_{n-1}$

③ $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

④ $(n+1)P_n = P'_{n+1} - xP'_n$

⑤ $(1-x^2)P'_n = n(P_{n-1} - xP_n)$

⑥ $(1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$

proof ① we know that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_{n(x)}$$

diff w.r.t z

$$-\frac{1}{2}(1-2xz+z^2)\cdot(-2x+2z) = \sum nz^{n-1}P_n$$

$$= (1-2xz+z^2)^{-3/2}(x-z) = \sum_{n=0}^{\infty} nz^{n-1}P_n$$

multiply by $(1-2xz+z^2)$

$$(1-2xz+z^2)^{-1/2}(x-z) = (1-2xz+z^2)\sum nz^{n-1}P$$

$$(x-z)\sum z^n P_n = (1-2xz+z^2)\sum nz^{n-1}P$$

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Bessel's diff^l eq^n.

A diff^l eq^n of the form

$x^2y'' + xy' + (x^2 - n^2)y = 0$ — (1) is known as Bessel's diff^l eq^n, where n is any non-negative constant.

above eq^n can also be written as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

sol^n: let $y = \sum_{r=0}^{\infty} a_r x^{m+r}$

$$y' = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

Substitute these values in eq^n (1)

$$x^2\left\{\sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}\right\} + x\left\{\sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}\right\} + (x^2 - n^2)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r} + \sum_{r=0}^{\infty} a_r (m+r) x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r \left\{(m+r)(m+r-1) + (m+r) - n^2\right\} x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r-2} = 0$$

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$$\sum_{r=0}^{\infty} a_r \left\{(m+r)(m+r-1+1) - n^2\right\} x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r \left\{(m+r)^2 - n^2\right\} x^{m+r} + \quad " \quad " = 0$$

$$\sum_{r=0}^{\infty} a_r \left\{(m+r+n)(m+r-n)\right\} x^{m+r} + \quad " \quad " = 0$$

Equating the lowest power of x i.e x^m to zero

$$a_0 \left\{(m+n)(m-n)\right\} = 0$$

$$\Rightarrow (m+n)(m-n) = 0 \quad \because a_0 \neq 0$$

$$\Rightarrow m = n, -n$$

Equating the coefficient of x^{m+1} to zero

$$a_1 \left\{(m+n+1)(m-n+1)\right\} = 0$$

$$a_1 = 0 \quad \text{or} \quad (m+n+1)(m-n+1) \neq 0$$

Equating the coefficient of x^{m+r}

$$a_r \left\{(m+r+n)(m+r-n)\right\} + a_{r-2} = 0$$

$$a_r = \frac{-1}{(m+r+n)(m+r-n)} a_{r-2}$$

Put $r = 2$

$$a_2 = \frac{-1}{(m+2+n)(m+2-n)} a_0$$

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Linear partial diff^{l} eqn of order one.

Lagranges equation: A partial diff^{l} eqn of the form $Pp + Qq = R$, where P, Q & R are functions of x, y & z, is called Lagrange's partial diff^{l} eqn of order one.

For ex.

① $x(y-z)p + y(z-x)q = z(x-y)$

② $xp + yq = z$

working rule:

① put the diff^{l} eqn in the Lagrange's form

② Find the value of P, Q & R

③ Substitute the value of P, Q & R in auxilliary eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

④ solve these auxillrary eqn

⑤ write the solution in following any one form

(a) $F(u, v) = 0$

(b) $u = f(v)$

(c) $v = f(u)$

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where u and v are solution and F are arbitrary function

Type - I

ex $\frac{y^2 z}{x} p + xzq = y^2$

Solⁿ: Compare with $Pp + Qq = R$

$P = \frac{y^2 z}{x}, Q = xz, R = y^2$

Then

$$A.E = \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

From fraction I & II

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$\Rightarrow x^2 dx - y^2 dy = 0$$

$$\frac{x^3}{3} - \frac{y^3}{3} = \frac{c_1}{3} \Rightarrow \boxed{x^3 - y^3 = c_1}$$

Solution of non-linear equation (PDE) Charpits method:

Working rules:

① write the given PDE in right side. ie $f(x, y, z, p, q) = 0$

② write down the A.E

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

where $F_x = \frac{\partial f}{\partial x}$

③ solve any two function for p & q,

④ put the values of p & q in
$dz = pdx + qdy$

⑤ by integrating $dz = pdx + qdy$, we get the solution

ex $z = px + qy + p^2 + q^2$

Sol^n $z - px - qy - p^2 - q^2 = 0$

Here, $F(x, y, z, p, q) = z - px - qy - p^2 - q^2$

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

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(11) Clairauts Form :-
A diff eqn of the form ~~z = px+q~~
$z = px + qy + F(p,q)$ is known as Clairauts form.

Solution:
If the given pde is form of Clairauts form then soln will be found by replacing $p=a$ & $q=b$ in given diff eqn

i.e $z = ax + by + F(a,b)$

ex
$z = px + qy + pq$

Soln: Given diff eqn is of the ~~form~~ Clairauts form then its solution will be found by replacing $p=a$ and $q=b$ in given diff eqn

$$\boxed{z = ax + by + ab}$$

solve, (1) $(px + qy - z)^2 = 1 + p^2 + q^2$

(2) $z = px + qy + c\sqrt{1 + p^2 + q^2}$

(3) $z = px + qy + \log(pq)$

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Homogeneous partial diff eqn.

A partial diff eqn of the form

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{--- (1)}$$

is called homogeneous partial diff eqn of degree n.

Ex $$2\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial x^3 y} - \frac{\partial^4 z}{\partial y^4} = 0$$

Notation let $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$ then, eqn (1) can be written as.

$$[A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n] z = f(x,y)$$

or $F(D, D')z = f(x,y)$

Working Rule of for complementry function (C.F).

(1) Let the homogeneous P.DE with homoginity n is $F(D \cdot D')z = F(x,y)$ is given then C.F. will be found by replacing $D = m$ and $D' = 1$ in $F(D, D') = 0$

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(1) If $F(m) = 0$ then it has either different real roots or equal real roots then its solution will be (For different roots $m_1 \neq m_2 \neq m_3 = \cdots m_n$)

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

If roots are equal then solution will be

$$z = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \cdots$$

(2) If $f(D, D') = F(D')$, then sol^n will be in form of $\phi(x)$

(3). If $F(D, D') = F(D)$ then solution will be in form of $\phi(y)$

Notation :-

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{dz}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

Short Method to find particular integral (P.I):

(1) If $F(D,D')$ be homogeneous function of degree n then P.I corresponding to

(a) $$\frac{1}{F(D,D')}\phi(ax+by)=\frac{1}{F(a,b)}\underbrace{\iiint\cdots\int}_{n\text{ times}} v\,dv$$

where $v=ax+by$

and $F(a,b)\neq 0$

(b) If $F(a,b)=0$ then

$$\frac{1}{F(a,b)}\phi(ax+by)=\frac{x^n}{b^n n!}\phi(ax+by)$$

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14 Particular Integral when $F(x,y) = x^m y^n$ where m & n are any integral.

For solution we expand the function $F(D, D')$ in binomial form ~~then~~ and then get the P.I.

Remark We can expand $F(D, D')$ in $\left(\frac{D}{D'}\right)$ if $m<n$ and $\left(\frac{D'}{D}\right)$ if $n<m$, x^m, y^n

Solve $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

Sol^n: $C.F = (m^3 - 1) = 0$

$\Rightarrow m = 1, w, w^2$

$$z = \phi_1(y+x) + \phi_2(y+wx) + \phi_3(y+w^2x)$$

$$PI = \frac{1}{F(D, D')} x^3 y^3$$

$$\frac{1}{D^3 - D'^3} x^3 y^3$$

$$= \frac{1}{D^3\left(1 - \frac{D'^3}{D^3}\right)} x^3 y^3$$

$$\frac{1}{D^3}\left(1 - \left(\frac{D'}{D}\right)^3\right)^{-1} x^3 y^3$$

$$\frac{1}{D^3}\left[1 + \left(\frac{D'}{D}\right)^3 + \left(\frac{D'}{D}\right)^6 \cdots\right] x^3 y^3$$

$$\frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3}D'^3(x^3y^3) + \frac{1}{D^6}D'^6(x^3y^3) + \cdots\right]$$

$$\frac{1}{D^3}\left[x^3y^3 + \frac{1}{6D^3}6x^3 + \cdot 0\right]$$

$$\frac{1}{D^3}(x^3y^3) + \frac{1}{D^6}(6x^3)$$

$$\frac{y^3 x^6}{6\cdot 5\cdot 4} + \frac{6x^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}$$

Complete solution will be

$$z = C.F + P.I$$

Non-Homogeneous partial diff eqn.

A partial diff eqn is know as non-homogeneous PDE if the order of all partial derivativa are not equal.

Working Rule:

Let the given non-homogeneous PDE be $F(D, D')z = f(x, y)$

① Factorize $F(D, D') = 0$ into linear factor.

②. Corresponding to each other non-repeated factor $(bD - aD' - c)$ the part of C.F is taken as $e^{\frac{cx}{b}}\phi(by+ax)$ if $b \neq 0$

③. Corresponding repeated to factor $(bD - aD' - c)^r$ then the part of C.F will be

$$e^{\frac{cx}{b}}\left[\phi_1(by+ax) + x\phi_2(by+ax) + x^2\phi_3(by+ax) + \cdots + x^{r-1}\phi_r(by+ax)\right]$$

④. Corrosponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F will be taken as $e^{-\frac{c(y)}{a}}\phi(by+ax)$, if $a \neq 0$

⑤. Corrosponding to repeated factor $(bD - aD' - c)^r$ then part of C.F will be

$$e^{-\frac{cy}{a}}\left[\phi_1(by+ax) + y\phi_2(by+ax) + y^2\phi_3(by+ax) + \cdots + y^{r-1}\phi_r(by+ax)\right]$$

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Solve. $(D^2 - D'^2 + D - D')z = 0$

Sol^n.

$D^2 - D'^2 + D - D' = 0$

$\Rightarrow (D-D')(D+D') + (D-D') = 0$

$(D-D')(D+D'+1) = 0$

Sol^n. corresponding to factor $(D-D')$

$= e^{\frac{0 \cdot x}{1}} [\phi_1(1y + 1 \cdot x)$

$= \phi_1(y+x)$

Sol^n corresponding to factor $(D+D'+1) = 0$

$e^{\frac{-1 \cdot x}{1}} \phi_2(1y - 1 \cdot x)$

$= e^{-x} \phi_2(y-x)$

Sol^n. is

$z = \phi_1(y+x) + e^{-x}\phi_2(y-x)$

Working Rule for Finding C.F of Irredcible non-homogeneous P.D.E.

Step-I: Take a trail Solution:

$$Z = Ae^{hx+ky}$$, where A a a constant

Step-II:

Find DZ, $D'Z$ -- and put it into given P.D.E

Step-III Find the value of h & K

Step-IV put the value of h or K in

$$Z = Ae^{hx+ky}.$$

Partial particular integral of non-homogeneous PDE.

Let the non-homogeneous PDE be $F(D, D') = f(x, y)$

(i) When $f(x, y) = e^{ax+by}$ and $F(a, b) \neq 0$

then $P.I = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$

(ii) When $f(x, y) = \sin(ax+by)$ or $\cos(ax+by)$ the P.I of $\frac{1}{F(D, D')} \sin(ax+by)$ can be found by putting $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$

(iii) When $f(x, y) = x^m y^n$

then $P.I = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$

(iv) When $f(x, y) = V e^{ax+by}$, where V is any function of x, y then

$P.I = \frac{1}{F(D, D')} V e^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V$

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Newton-Raphson Method : Let $f(x)=0$ is an equation which is defined in the interval $[a, b]$. Then Newton-Raphson Method is defined as.

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

where x_k is .. k^{th} iterati

for ex. Find the real root of the eqn $x^2 + 4\sin x = 0$, which is closure to $x = -1.9$, by using Newton-Raphson formula.

Sol^n: Given that

$$f(x) = x^2 + 4\sin x \quad , \quad x_0 = -1.9$$

$$f'(x) = 2x + 4\cos x$$

then by Newton Raphson folumla

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{(x_k^2 + 4\sin(x_k))}{2x_k + 4\cos(x_k)}$$

$$x_{k+1} = \frac{x_k^2 + 4x_k\cos(x_k) - 4\sin(x_k)}{2x_k + 4\cos(x_k)}$$

Application of Newton-Raphson Method.

(1) Find the square roots of any number:

Solution: Let α be a number, whose square roots to be find.

Let $x = \sqrt{\alpha}$

$\Rightarrow x^2 = \alpha$

Let $f(x) = x^2 - \alpha$, $f'(x) = 2x$

$\therefore$ By Newton Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^2 - \alpha)}{2x_n} = \frac{2x_n^2 - x_n^2 + \alpha}{2x_n}$$

$$= \frac{x_n^2 + \alpha}{2x_n}$$

ex. Find the square root of the no. 32

Soln. Let $x = \sqrt{32}$

$\Rightarrow x^2 - 32 = 0$

By Newton Raphson method.

$$x_{n+1} = x_n - \frac{x_n^2 - 32}{2x_n}$$

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$$x_{n+1} = \frac{2x_n^2 - x_n^2 + 34}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 34}{2x_n}$$

Let, $x_0 = 5.5$

$$x_1 = \frac{x_0^2 + 34}{2 \cdot x_0^2}$$

$$\frac{30.25 + 34}{2 \times 5.5}$$

$$x_1 \quad \frac{64.25}{11} = 5.84$$