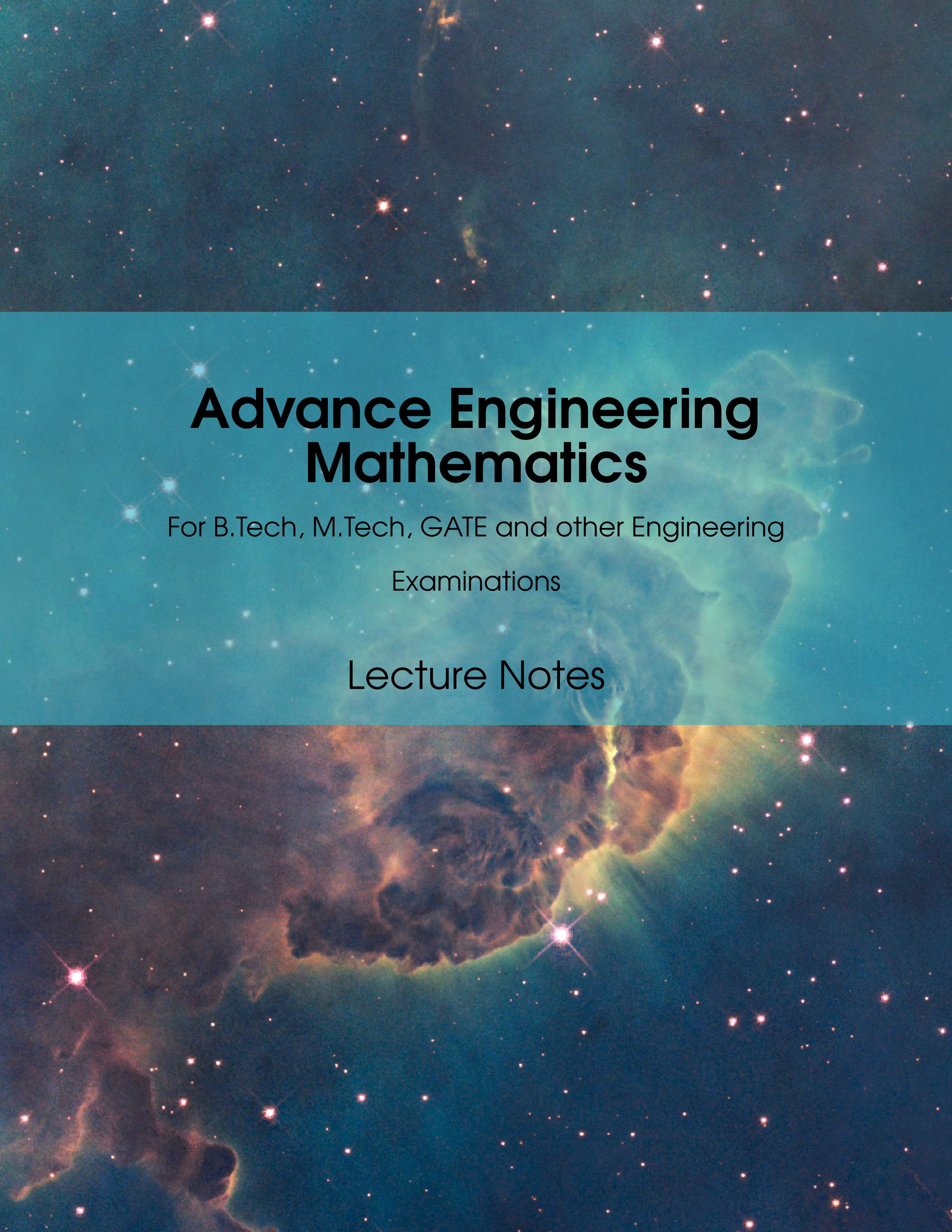
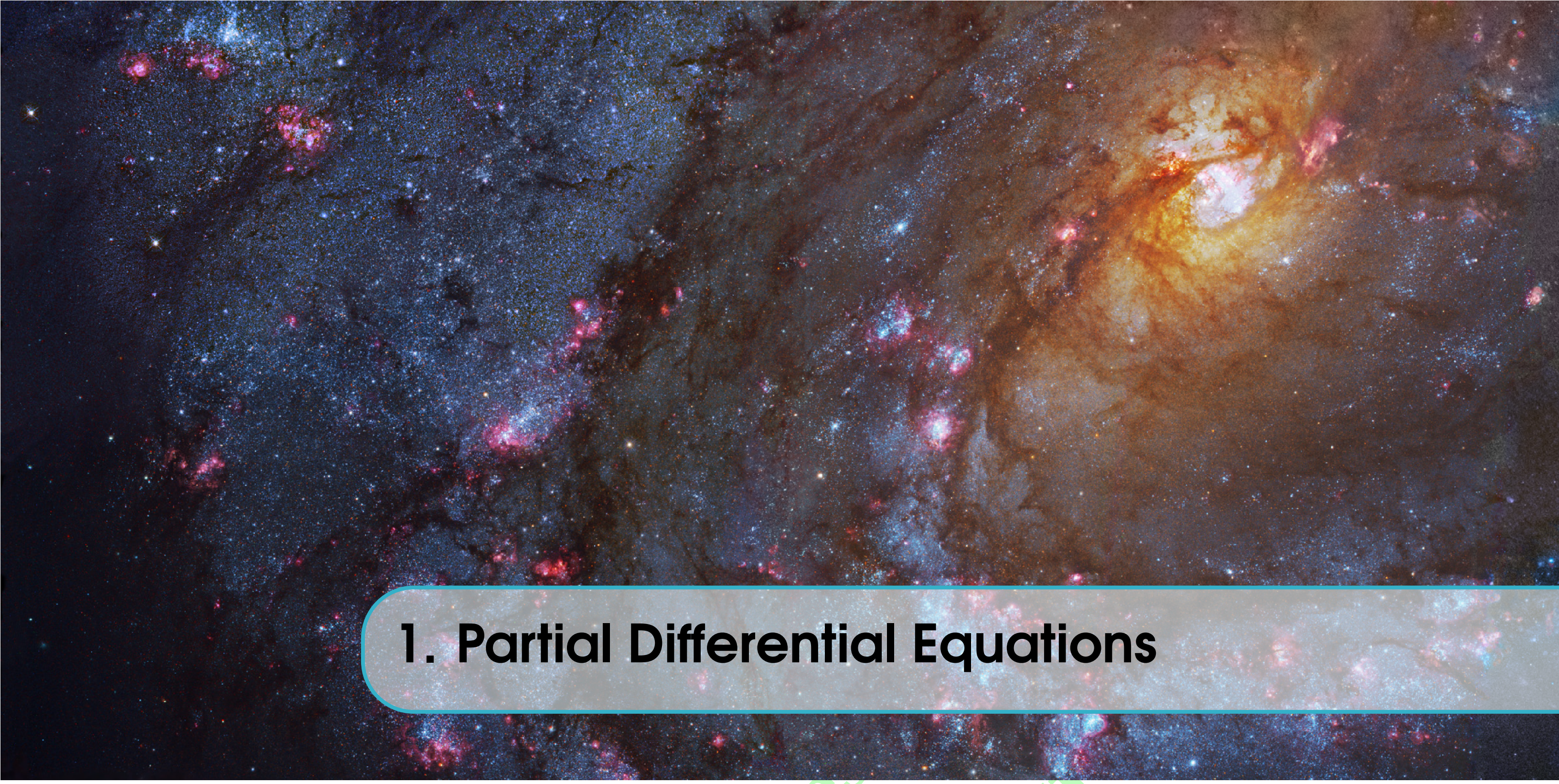


Advance Engineering Mathematics

For B.Tech, M.Tech, GATE and other Engineering Examinations

Lecture Notes

Contents



1. Partial Differential Equations

1.1 Partial Differential Equation (P.D.E.)

We will be studying functions $z = z(x^1, x^2, \ldots, x^n)$ and their partial derivatives. Here $x^1, x^2, \ldots, x^n$ are standard Cartesian coordinates on $\mathbb{R}^n$. We sometimes use the alternate notation $u(x,y), u(x,y,z)$, etc. We also write e.g. $z(r,\theta,\phi)$ for spherical coordinates on $\mathbb{R}^3$, etc.

Notation 1.1. *Let us consider a function $u(x,y)$ of two independent variables x and y. We use the following notation for partial derivatives:*

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}$$

Definition 1.1.1 An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a **partial differential equation**.

Example 1.1 The example of PDE are as follows:

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial y^2} = 2xy\frac{\partial z}{\partial x}$

(iii) $p + q = y^2$

(vi) $r^3 + 2s + t^3 = 0$

■

Definition 1.1.2 The **order** of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation. For example: The order of examples (i) and (iii) are 1 while the order of examples (ii) and (iv) are 2.

Definition 1.1.3 The **degree** of a partial differential equation is defined as the degree of the highest order partial derivative occurring in the partial differential equation after the equation has been rationalized i.e. made free from redicals and fractions. For example: The degree of examples (i), (ii) and (iii) is 1 while the degree of example (iv) is 3.

Definition 1.1.4 A partial differential equation is said to be **linear** if the dependent variable and its partial derivatives occur only in first degree and are not multiplied. A partial differential equation which is not linear is called a **non-linear** partial differential equation. For example: The examples (i) and (iii) are linear while the examples (ii) and (iv) are non-linear.

1.1.1 Solution of first order Partial Differential Equations

Lagrange's partial differential equations of first order: A partial differential equation of the form $Pp+Qq=R$ is called Lagrange's partial differential equations of first order, where P, Q, R are functions of x,y,z only and $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$.

Working Rule: To solve the Lagrange's PDE follow the following steps:
Step-I: Rewrite the given PDE into standard form $Pp+Qq=R$.
Step-II: Write down the auxiliary as

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

Step-III: By taking any two fraction, solve the above auxiliary equation. Let the two solution be $u=c_1$ and $v=c_2$.
Step-IV: Write the solution in following any one form:
$f(u,v)=0$ or $u=f(v)$ or $v=f(u)$, where f is any arbitrary function.

■ **Example 1.2** Solve the partial differential equation $yp+yq=z^2+1$. ■

Solution: By comparing with the first order Lagrange's partial differential equation $Pp+Qq=R$, we get $P=y$, $Q=y$ and $R=z^2+1$. The auxiliary equations are

$$\frac{dx}{y}=\frac{dy}{y}=\frac{dz}{z^2+1}.$$

By taking first two fraction, we get $\frac{dx}{y}=\frac{dy}{y}\implies dx=dy$
By integrating, we have

$$x=y+c_1 \implies x-y=c_1 \tag{1.1}$$

By taking last two fraction, we get $\frac{dy}{y}=\frac{dz}{z^2+1}$
By integrating, we have

$$\log y=\tan^{-1}z+c_2 \implies \log y-\tan^{-1}z=c_2 \tag{1.2}$$

From equations (1.1) and (1.2), we can write the solutions in following any one form:
$f(x-y,\log y-\tan^{-1}z)=0$ or $x-y=f(\log y-\tan^{-1}z)$ or $\log y-\tan^{-1}z=f(x-y)$. □

Type-1

■ **Example 1.3** Solve the partial differential equation $y^2p-xyq=x(z-2y)$. ■

Solution: By comparing with the first order Lagrange's partial differential equation $Pp+Qq=R$, we get $P=y^2$, $Q=-xy$ and $R=x(z-2y)$. The auxiliary equations are

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}.$$

By taking first two fraction, we get

$$\frac{dx}{y^2}=\frac{dy}{-xy} \implies \frac{dx}{y}=\frac{dy}{-x} \implies xdx+ydy=0$$

By integrating, we have

$$x^2+y^2=c_1 \tag{1.3}$$

By taking last two fraction, we get

$$\begin{aligned}\frac{dy}{-y}=\frac{dz}{(z-2y)} &\implies (z-2y)dy+ydz=0\\ &\implies zdy-2ydy+ydz=0\\ &\implies zdy+ydz=2ydy\\ &\implies d(yz)=2ydy\end{aligned}$$

By integrating, we have

$$yz=y^2+c_2 \implies yz-y^2=c_2 \tag{1.4}$$

From equations (1.3) and (1.4), the solutions is $f(x^2+y^2, yz-y^2)=0$. □

Exercise

Solve the following PDE:

(1) $xp+yq=z$ Ans. $f(x/z, y/z)=0$
(2) $p+q=1$ Ans. $f(x-y, x-z)=0$
(3) $x^2p+y^2q+z^2=0$ Ans. $f(\frac{1}{x}-\frac{1}{y}, \frac{1}{y}+\frac{1}{z})=0$
(4) $yzp+zxq=xy$ Ans. $f(x^2-z^2, x^2-y^2)=0$
(5) $zp=x$ Ans. $f(x^2-z^2, y)=0$

Type-2 (Substitution)

■ **Example 1.4** Solve the partial differential equation xzp + yzq=xy. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy}.$$

Taking first two fraction, we get

$$\frac{dx}{x}=\frac{dy}{y}.$$

By integrating, we have

$$\log x=\log y+\log c_1 \implies x=c_1y. \tag{1.5}$$

Now taking last two fraction and by putting $x=c_1y$ in last fraction, we have

$$\frac{dy}{z} = \frac{dz}{c_1 y}.$$

By integrating, we have

$$\frac{c_1 y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2} \implies c_1 y^2 = z^2 + c_2. \tag{1.6}$$

By putting the value of $c_1 = x/y$ from equation (1.5), we have

$$xy - z^2 = c_2 \tag{1.7}$$

From equations (1.5) and (1.7) the required solution is $f(x/y, xy - z^2) = 0$ □

■ **Example 1.5** Solve the partial differential equation $p + aq = z + \cot(y - ax)$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{a} = \frac{dz}{z + \cot(y - ax)}.$$

Taking first two fraction, we get

$$\frac{dx}{1} = \frac{dy}{a}.$$

By integrating, we have

$$ax = y - c_1 \implies y - ax = c_1. \tag{1.8}$$

Now taking last two fraction and by putting $y - ax = c_1$ in last fraction, we have

$$\frac{dy}{a} = \frac{dz}{z + \cot c_1}.$$

By integrating, we have

$$y/a = \log(z + \cot c_1) + c_2. \tag{1.9}$$

By putting the value of $c_1 = y - ax$ from equation (1.8), we have

$$y/a - \log(z + \cot(y - ax)) = c_2 \tag{1.10}$$

From equations (1.8) and (1.10) the required solution is $f(y - ax, y/a - \log(z + \cot(y - ax))) = 0$
□

■ **Example 1.6** Solve the partial differential equation $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$. ■

Solution The given equation can re-written as $x(z - 2y^2)p + y(z - y^2 - 2x^3)q = z(z - y^2 - 2x^3)$. The Lagrange's auxiliary equations are

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} = \frac{dz}{z(z - y^2 - 2x^3)}.$$

Taking last two fraction, we get

$$\frac{dy}{y} = \frac{dz}{z}.$$

By integrating, we have

$$\log y = \log z - \log c_1 \implies c_1 y = z. \tag{1.11}$$

Now taking first two fraction and by putting $z = c_1 y$ in both fraction, we have

$$\frac{dx}{x(c_1y-2y^2)}=\frac{dy}{y(c_1y-y^2-2x^3)}.$$

$$\implies y(c_1y-y^2-2x^3)dx = x(c_1y-2y^2)dy \implies (c_1y-y^2-2x^3)dx = x(c_1-2y)dy$$

$$(c_1y-y^2-2x^3)dx + x(2y-c_1)dy = 0. \tag{1.12}$$

which is of the form $Mdx+Ndy=0$. Here $M=c_1y-y^2-2x^3$ and $N=x(2y-c_1)$. Then $\partial M/\partial y = c_1-2y$ and $\partial N/\partial x = 2y-c_1$. Now we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x(2y-c_1)}\times 2(c_1-2y) = -\frac{2}{x}$$

which is function of x alone. Hence by usual rule, integrating factor will be $e^{\int(-2/x)dx}=e^{-2\log x}=x^{-2}$. Multiply the the equation (1.12) by integrating factor I.F.$=x^{-2}$, we have $x^{-2}(c_1y-y^2-2x^3)dx+x^{-1}(2y-c_1)dy=0$, which must be exact differential equation. Hence its solution is

$$\int\left\{x^{-2}(c_1y-y^2-2x^3)\right\}dx+\int x^{-1}(2y-c_1)=c_2 \tag{1.13}$$

(treating y as a constant) (terms free from x)

$$(c_1y-y^2)\times(-1/x)-x^2=c_2 \implies \frac{(y^2-c_1y)}{x}-x^2=c_2. \tag{1.14}$$

By putting the value of $c_1=z/y$ from equation (1.11) in equation (1.14), we have

$$\frac{(y^2-z)}{x}-x^2=c_2 \tag{1.15}$$

From equations (1.11) and (1.15) the required solution is $f\left(\frac{z}{y},\frac{(y^2-z)}{x}-x^2\right)=0$ □

Exercise

Solve the following PDE:

(1) $p-2q=3x^2\sin(y+2x)$ Ans. $f(x^2\sin(y+2x)-z, y+2x)=0$
(2) $p-q=z/(x+y)$ Ans. $f(xy, \log z+(ax/3y^2))=0$
(3) $(x^2-y^2-z^2)p+2xyq=2xz$ Ans. $f((x^2+y^2+z^2)/z, y/z)=0$
(4) $zp-zq=x+y$ Ans. $f(2x(x+y)-z^2, x+y)=0$

Type-3 (Multiplier Method)

■ **Example 1.7** Solve the partial differential equation $(mz-ny)p+(nx-lz)q=ly-mx$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}.$$

Using multipliers l,m,n, each fraction becomes

$$=\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}.$$

$\implies ldx+mdy+ndz=0$. By integrating we get

$$lx+my+nz=c_1. \tag{1.16}$$

Now, again using multipliers x,y,z, each fraction becomes

$$=\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}.$$

$\implies xdx+ydy+zdz=0$. By integrating we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_2}{2} \implies x^2+y^2+z^2=c_2. \tag{1.17}$$

From equations (1.16) and (1.17) the required solution is $f(lx+my+nz,x^2+y^2+z^2)=0$, □

■ **Example 1.8** Solve the partial differential equation $z(x+y)p+z(x-y)q=x^2+y^2$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{x^2+y^2}.$$

Using multipliers $x,-y,-z$, each fraction becomes

$$=\frac{xdx-ydy-zdz}{xz(x+y)-yz(x-y)-z(x^2+y^2)}=\frac{xdx-ydy-zdz}{0}.$$

$\implies xdx-ydy-zdz=0$. By integrating we get

$$x^2-y^2-z^2=c_1. \tag{1.18}$$

Again, using multipliers $y,x,-z$, each fraction becomes

$$=\frac{ydx+xdy-zdz}{yz(x+y)+xz(x-y)-z(x^2+y^2)}=\frac{ydx+xdy-zdz}{0}.$$

$\implies ydx+xdy-zdz=0 \implies d(xy)-zdz=0$. By integrating we get

$$xy-z^2/2=c_2. \tag{1.19}$$

From equations (1.18) and (1.19) the required solution is $f(x^2-y^2-z^2,xy-z^2/2)=0$, □

■ **Example 1.9** Solve the partial differential equation $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}.$$

Using multipliers $x,y,-1$, each fraction becomes

$$=\frac{xdx+ydy-dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}=\frac{xdx+ydy-dz}{0}.$$

$\implies xdx + ydy - dz = 0$. By integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2} \implies x^2 + y^2 - 2z = c_1. \tag{1.20}$$

Again, using multipliers $1/x, 1/y, 1/z$, each fraction becomes

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{(y^2+z) - (x^2+z) + (x^2-y^2)} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}.$$

$\implies (1/x)dx + (1/y)dy + (1/z)dz = 0$. By integrating we get

$$\log x + \log y + \log z = \log c_2 \implies xyz = c_2. \tag{1.21}$$

From equations (1.20) and (1.21) the required solution is $f(x^2+y^2-2z, xyz) = 0$, □

Exercise

Solve the following PDE:

(1) $(y-z)p + (z-x)q = x-y$ Ans. $f(x+y+z, x^2+y^2+z^2) = 0$
(2) $(y+zx)p - (x+yz)q + y^2 - x^2 = 0$ Ans. $f(xy+z, x^2+y^2-z^2) = 0$
(3) $x(y-z)p + y(z-x)q = z(x-y)$ Ans. $f(x+y+z, xyz) = 0$
(4) $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$ Ans. $f(x^2+y^2+z^2, x/yz) = 0$
(5) $(y-zx)p + (x+yz)q = x^2+y^2$ Ans. $f(x^2-y^2+z^2, xy-z) = 0$

Type-4

Example 1.10 Solve the partial differential equation $(y+z)p + (z+x)q = x+y$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \tag{1.22}$$

Using multipliers $1, -1, 0$, each fraction of (1.22) becomes

$$= \frac{dx - dy}{(y+z) - (z+x)} = \frac{d(x-y)}{-(x-y)}. \tag{1.23}$$

Again, using multipliers $0, 1, -1$, each fraction of (1.22) becomes

$$= \frac{dy - dz}{(z+x) - (x+y)} = \frac{d(y-z)}{-(y-z)} \tag{1.24}$$

Finally, using multipliers $1, 1, 1$, each fraction of (1.22) becomes

$$= \frac{dx + dy + dz}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \tag{1.25}$$

Equations (1.23, 1.24 and 1.25)

$$\implies \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}. \tag{1.26}$$

Taking the first two fraction of (1.26), we get

$$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

By integrating, we get

$$\log(x-y) = \log(y-z) + \log c_1 \implies \frac{(x-y)}{(y-z)} = c_1. \tag{1.27}$$

Taking the last two fraction of (1.26), we get

$$\frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \implies 2\frac{d(y-z)}{(y-z)} + \frac{d(x+y+z)}{(x+y+z)} = 0$$

By integrating, we get

$$2\log(y-z) + \log(x+y+z) = \log c_2 \implies (y-z)^2(x+y+z) = c_2. \tag{1.28}$$

From equations (1.27) and (1.28) the required solution is $f\left(\frac{(x-y)}{(y-z)}, (y-z)^2(x+y+z)\right) = 0.$
□

■ **Example 1.11** Solve the partial differential equation $(x^2-yz)p + (y^2-zx)q = z^2-xy$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \tag{1.29}$$

Using multipliers $1, -1, 0$, each fraction of (1.29) becomes

$$= \frac{dx-dy}{(x^2-yz)-(y^2-zx)} = \frac{d(x-y)}{x^2-y^2+z(x-y)} \implies \frac{d(x-y)}{(x-y)(x+y+z)}. \tag{1.30}$$

Again, using multipliers $0, 1, -1$, each fraction of (1.29) becomes

$$= \frac{dy-dz}{(y^2-zx)-(z^2-xy)} = \frac{d(y-z)}{y^2-z^2+x(y-z)} \implies \frac{d(y-z)}{(y-z)(y+z+x)} \tag{1.31}$$

Once again, using multipliers x, y, z, each fraction of (1.29) becomes

$$= \frac{xdx+ydy+zdz}{x(x^2-yz)+y(y^2-zx)+z(z^2-xy)} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$\implies \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \tag{1.32}$$

Finally, using multipliers $1, 1, 1$, each fraction of (1.29) becomes

$$\frac{dx+dy+dz}{(x^2+y^2+z^2-xy-yz-zx)} \tag{1.33}$$

Equations (1.30), (1.31), (1.32) and (1.33)

$$\begin{aligned}\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(y+z+x)} &= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \\ &= \frac{dx+dy+dz}{(x^2+y^2+z^2-xy-yz-zx)}.\end{aligned} \tag{1.34}$$

Taking the first two fraction of (1.34), we get

$$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

By integrating, we get

$$\log(x-y) = \log(y-z) + \log c_1 \implies \frac{(x-y)}{(y-z)} = c_1. \tag{1.35}$$

Taking the last two fraction of (1.34), we get

$$\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{(x^2+y^2+z^2-xy-yz-zx)}$$

$$\implies \frac{xdx+ydy+zdz}{(x+y+z)} = dx+dy+dz$$

$$\implies xdx+ydy+zdz-(x+y+z)d(x+y+z) = 0$$

By integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}-\frac{(x+y+z)^2}{2} = \frac{c_2}{2} \implies (x^2+y^2+z^2)-(x+y+z)^2 = c_2. \tag{1.36}$$

From equations (1.35) and (1.36) the solution is $f\left(\frac{(x-y)}{(y-z)}, (x^2+y^2+z^2)-(x+y+z)^2\right) = 0$
□

■ **Example 1.12** Solve the partial differential equation $\cos(x+y)p+\sin(x+y)q = z$. ■

Solution The Lagrange's auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \tag{1.37}$$

Using multipliers $1,1,0$, each fraction of (1.37) becomes

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \tag{1.38}$$

Again, using multipliers $1,-1,0$, each fraction of (1.37) becomes

$$= \frac{dx-dy}{\cos(x+y)-\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)} \tag{1.39}$$

Equations (1.37), (1.38) and (1.39)

$$\implies \frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)} \tag{1.40}$$

Taking the first two fraction of (1.40), we get

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} \tag{1.41}$$

Let $x+y=t$ so that $d(x+y)=dt$. Then second fraction of the above equation can be written as

$$\frac{dt}{\cos t+\sin t}=\frac{dt}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t\right)}=\frac{dt}{\sqrt{2}\,(\sin\pi/4\cos t+\cos\pi/4\sin t)}$$

$$=\frac{dt}{\sqrt{2}\,(\sin(t+\pi/4))}$$

Thus from equation (1.41), we have

$$\frac{dz}{z}=\frac{dt}{\sqrt{2}\,(\sin(t+\pi/4))}\implies\sqrt{2}\frac{dz}{z}=cosec(t+\pi/4)dt$$

By integrating, we get

$$\sqrt{2}\log z=\log\tan\frac{1}{2}(t+\pi/4)+\log c_1\implies z^{\sqrt{2}}=c_1\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$$
$$\implies z^{\sqrt{2}}\cot\left(\frac{t}{2}+\frac{\pi}{8}\right)=c_1$$
$$\implies z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right)=c_1\qquad\because t=x+y.\tag{1.42}$$

Taking the last two fraction of (1.40), we get

$$\frac{d(x-y)}{\cos(x+y)-\sin(x+y)}=\frac{d(x+y)}{\cos(x+y)+\sin(x+y)}$$

$$d(x-y)=\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)\tag{1.43}$$

Let $x+y=t$ so that $d(x+y)=dt$. Then the above equation can be written as

$$d(x-y)=\frac{\cos t-\sin t}{\cos t+\sin t}dt$$

By integrating, we get

$$x-y=\log(\sin t+\cos t)-\log c_2\implies\frac{(\sin t+\cos t)}{c_2}=e^{x-y}\implies e^{-(x-y)}(\sin t+\cos t)=c_2$$

$$=e^{-(x-y)}(\sin t+\cos t)=c_2\implies e^{-(x-y)}(\sin(x+y)+\cos(x+y))=c_2\tag{1.44}$$

From equations (1.42) and (1.44) the required solution is

$$f\left(z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right),e^{-(x-y)}(\sin(x+y)+\cos(x+y))\right)=0.$$

□

Exercise

Solve the following PDE:

(1) $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$ Ans. $f(\frac{x-y}{z},x^3+y^3)=0$

(2) $(x^2-y^2-z^2)p+2xyq=2xz$ Ans. $f(\frac{y}{z},\frac{x^2+y^2+z^2}{z})=0$

(3) $(1+y)p+(1+x)q=z$ Ans. $f((1+x)^2-(1+y)^2,(2+x+y)/z)=0$

(4) $(x^2+y^2+yz)p+(x^2+y^2-xz)q=z(x+y)$ Ans. $f((x^2+y^2)/z^2,z-x+y)=0$

(5) $p+q=x+y+z$ Ans. $f(x-y,e^{-x}(2+x+y+z))=0$

1.1.2 General method of solving partial differential equations of order one but of any degree (non-linear)

Charpit's Method

Working Rule: Let the given partial differential equation of first order and non-linear in p and q be$f(x,y,z,p,q)=0$

Step-I. Transfer all the terms of the given equation to L.H.S. and denote the entire expression by f.

Step-II. Write the Charpit's auxiliary equations as follows:

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0}$$

Where $f_x=\frac{\partial f}{\partial x},\qquad f_y=\frac{\partial f}{\partial y},\qquad f_p=\frac{\partial f}{\partial p},\ldots$

Step-III. Put the value of f_x, f_y, f_p,... etc, in the Charpit's auxiliary equation.

Step-IV. Choose two proper fraction from the above auxiliary equation so that we can integrate them easily and find the value of p and q.

Step-V. Put the values of p and q in $dz=pdx+qdy$. By integrating this we get the complete integral of the given equations.

■ **Example 1.13** Find the complete integral of $z=px+qy+p^2+q^2$. ■

Solution Let $f(x,y,z,p,q)=z-px+qy+p^2+q^2=0$. The Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dF}{0} \tag{1.45}$$

Here $f_x=-p,\qquad f_y=-q,\qquad f_z=1,\qquad f_p=-x-2p$ and $f_q=-y-2q$

Put all these values in equation (1.45), we have

$$\frac{dp}{(-p)+p(1)}=\frac{dq}{(-q)+q(1)}=\frac{dz}{-p(-x-2p)-q(-y-2q)}=\frac{dx}{-(-x-2p)}=\frac{dy}{-(-y-2q)}=\frac{dF}{0}$$

The above equations reduces to

$$\frac{dp}{0}=\frac{dq}{0}=\frac{dz}{p(x+2p)+q(y+2q)}=\frac{dx}{(x+2p)}=\frac{dy}{(y+2q)}=\frac{dF}{0} \tag{1.46}$$

The first fraction of (1.46) $\Longrightarrow$ $dp=0$ so that $p=a($ where a is an arbitrary constant)

Similarly, second fraction of (1.46) $\Longrightarrow$ $dq=0$ so that $q=b($ where b is an arbitrary constant)

Putting the value of $p=a$ and $q=b$ in the given equation $z=px+qy+p^2+q^2$, we get the required integral as $z=ax+by+a^2+b^2$. □

Example 1.14 Find the complete integral of $zpq = p + q$. ∎

Solution Let $f(x,y,z,p,q) = zpq - p + q = 0$. The Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \tag{1.47}$$

Here $f_x = 0, \quad f_y = 0, \quad f_z = pq, \quad f_p = zq - 1$ and $f_q = zp - 1$
Put all these values in equation (1.47), we have

$$\frac{dp}{(0) + p(pq)} = \frac{dq}{(0) + q(pq)} = \frac{dz}{-p(zq-1) - q(zp-1)} = \frac{dx}{-(zq-1)} = \frac{dy}{-(zp-1)} = \frac{dF}{0}$$

The above equations reduces to

$$\frac{dp}{p^2q} = \frac{dq}{pq^2} = \cdots \implies \frac{dp}{p} = \frac{dq}{q} \tag{1.48}$$

By integrating we get, $\log p = \log q + \log a \implies p = aq$. Put the value of $p = aq$ in given equation $zpq = p + q$, we get the value of $q = \frac{(1+a)}{az}$ and $p = \frac{(1+a)}{z}$. Putting the value of $p = \frac{(1+a)}{z}$ and $q = \frac{(1+a)}{az}$ in the equation $dz = px + qy$, we get

$$dz = \frac{(1+a)}{z}dx + \frac{(1+a)}{az}dy \implies zdz = (1+a)dx + \frac{(1+a)}{a}dy.$$

By integrating we get the required integral as

$$\frac{z^2}{2} = (1+a)x + \frac{(1+a)}{a}y + b \implies z^2 = 2\left[(1+a)x + \frac{(1+a)}{a}y + b\right].$$

□

Exercise

Solve the following PDE:

(1) $px + qy = pq$ Ans. $az = \frac{(ax+y)^2}{2} + b$
(2) $p^2x + q^2y = z$ Ans. $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$
(3) $2z + p^2 + qy + 2y^2 = 0$ Ans. $2y^2z + y^2(x-a)^2 + y^4 = b$
(4) $z^2 = pqxy$ Ans. $z = x^a y^{1/a} b$
(5) $q = (z+px)^2$ Ans. $xz = 2\sqrt{ax} + ay + b$

Special Method to solve non-linear first order partial differential equations

Standard Form-I:(When PDE contains only p and q) Let the given equation which contains only p and q is $f(p,q) = 0$.
Step-I: Put the value $p = a$ and $q = b$ in the equation $dz = pdx + qdy$.
Step-II: By integrating the equation $dz = adx + bdy$ we get $z = ax + by + c$, where c is an integrating constant.
Step-III: Now put the value of $a = F(b)$ or $b = F(a)$ from given equation $f(a,b) = 0$ in $z = ax + by + c$
Step-IV: The required answer will be either $z = ax + F(a)y + c$ or $z = F(b)x + by + c$

Example 1.15 Solve $p^2+q^2=m^2$

Solution: Given PDE is

$$p^2+q^2=m^2 \tag{1.49}$$

Which is of the form $f(p,q)=0$. Therefore its solution can be found by putting $p=a$ and $q=b$ in the equation

$$dz=pdx+qdy \quad i.e. \quad dz=adx+bdy \tag{1.50}$$

By integrating we get

$$z=ax+by+c \tag{1.51}$$

Also from equation (1.49), we have

$$a^2+b^2=m^2 \implies a=\sqrt{m^2-b^2}$$

Thus the required solution will be

$$z=\sqrt{m^2-b^2}x+by+c$$

Example 1.16 Find the solution of $z^2p^2y+6zpxy+2zqx^2+4x^2y=0$

Solution: The given equation can be written as

$$z^2\left(\frac{\partial z}{\partial x}\right)^2 y+6z\left(\frac{\partial z}{\partial x}\right)xy+2z\left(\frac{\partial z}{\partial y}\right)x^2+4x^2y=0 \tag{1.52}$$

By dividing x^2y, we get

$$\left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2+6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right)+2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right)+4=0 \tag{1.53}$$

Let $xdx=dX \quad ydy=dY$ and $zdz=dZ$ so that $x^2/2=X \quad y^2/2=Y$ and $z^2/2=Z$ Now equation (1.53) becomes

$$P^2+6P+2Q+4=0 \tag{1.54}$$

where $P=\dfrac{\partial Z}{\partial X}, Q=\dfrac{\partial Z}{\partial Y}$.
Which is of the form $f(P,Q)=0$. Therefore its solution can be found by putting $P=a$ and $Q=b$ in the equation

$$dZ=PdX+QdY \quad i.e. \quad dZ=adX+bdY \tag{1.55}$$

By integrating we get

$$Z=aX+bY+c \tag{1.56}$$

Also from equation (1.54), we have

$$a^2 + 6a + 2b + 4 = 0 \implies b = \frac{a^2 + 6a - 4}{2}$$

Thus the solution will be

$$Z = aX + \frac{a^2 + 6a - 4}{2} Y + c$$

By putting the value $X = x^2/2, \quad Y = y^2/2$ and $Z = z^2/2$, we get the required solution as

$$\frac{z^2}{2} = a\frac{x^2}{2} + \frac{a^2 + 6a - 4}{2}\frac{y^2}{2} + c \implies z^2 = ax^2 + \frac{a^2 + 6a - 4}{2} y^2 + 2c$$

Exercise

Solve the following PDE:

(1) $\sqrt{p} + \sqrt{q} = 1$ Ans. $z = ax + (1 - \sqrt{a})y + c$
(2) $pq = 1$ Ans. $z = ax + (1/a)y + c$
(3) $x^2p^2 + y^2q^2 = z$ Ans. $2\sqrt{z} = a\log x + \sqrt{(1 - a^2}\log y + c$ **Hint:** Take $(1/x)dx = dX$ $(1/y)dy = dY$ and $(1/\sqrt{z})dz = dZ$
(4) $z^2 = pqxy$ Ans. $z = x^a y^{1/a} C$ **Hint:** Take $(1/x)dx = dX$ $(1/y)dy = dY$ and $(1/z)dz = dZ$

Special Method to solve non-linear first order partial differential equations

Standard Form-II:(When PDE contains only p, q and z) Let the given equation which contains only p, q and z is $f(p,q,z) = 0$.
Step-I: Let $u = x + ay$ where a is any arbitrary constant.
Step-II: Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively. Solve the resulting ordinary differential equation of first order by usual methods
Step-III: Replace u by $x + ay$ in the solution obtained in step-II.

■ **Example 1.17** Solve $p^3 + q^3 - 3pqz$ ■

Solution: Given PDE is

$$p^3 + q^3 - 3pqz \tag{1.57}$$

Which is of the form $f(p,q,z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Therefore its solution can be found by putting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in the equation

$$p^3 + q^3 - 3pqz \implies \left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 - 3\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right)z \tag{1.58}$$

$$\implies (1 + a^3)\frac{dz}{du} = 3az \implies (1 + a^3)\frac{dz}{z} = 3a du \tag{1.59}$$

By integrating we get

$$(1+a^3)\log z = 3au + b \implies \log z - \log b = au \tag{1.60}$$

Thus the required solution is

$$\implies (1+a^3)\log z = 3a(x+ay)+b$$

Example 1.18 Solve $p^2 = qz$

Solution: Given PDE is

$$p^2 = qz \tag{1.61}$$

Which is of the form $f(p,q,z)=0$. Let $u=x+ay$, where a is an arbitrary constant. Therefore its solution can be found by putting $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$ in the equation

$$p^2 = qz \implies \left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \implies \frac{dz}{du} = az \implies \frac{dz}{z} = adu \tag{1.62}$$

By integrating we get

$$\log z = au + \log b \implies \log z - \log b = au \implies \frac{z}{b} = e^{au} \tag{1.63}$$

Thus the required solution is

$$z = be^{a(x+ay)}$$

Exercise

Solve the following PDE:

(1) $9(p^2z+q^2)=4$ **Ans.** $(x+ay+b)^2=(z+a^2)^3$
(2) $p(1+q^2)=q(z-\alpha)$ **Ans.** $(x+ay+b)^2=4\{a(z-\alpha)-1\}^2$
(3) $q^2=z^2p^2(1-p^2)$ **Ans.** $(x+ay+b)^2=(z^2-a^2)$
(4) $z^2(p^2z^2+q^2)=1$ **Ans.** $9(x+ay+b)^2=(z^2+a^2)^3$
(5) $4(1+z^3)=9z^4pq$ **Ans.** $(x+ay+b)^2=a(1+z^3)$

Special Method to solve non-linear first order partial differential equations

Standard Form-III:(When PDE contains only p, q, x and y) Let the given equation which contains only p, q, x and y is $f(p,q,x,y)=0$.
Step-I: Separate x, p one side and y, q one side, say $f_1(x,p)=f_2(y,q)$.
Step-II: Take $f_1(x,p)=f_2(y,q)=a(constant)$. Now consider $f_1(x,p)=a(constant)$ and $f_2(y,q)=a(constant)$
Step-III: Let $f_1(x,p)=a$ solve it for p, say $p=F_1(x,a)$. Similarly take $f_2(y,q)=a$ and solve it for q, say $q=F_2(y,a)$.
Step-IV: Put the value of $p=F_1(x,a)$ and $q=F_2(y,a)$ in the equation $dz=pdx+qdy$.
Step-V: The required solution will be $z=\int F_1(x,a)dx+\int F_2(y,a)dy+b$

■ **Example 1.19** Find the integral of $x(1+y)p = y(1+x)q$ ■

Solution: Separating p and x from q and y, the given equation can be written as

$$\frac{xp}{(1+x)} = \frac{yq}{(1+y)} \tag{1.64}$$

Equating each side to an arbitrary constant a, we get $\dfrac{xp}{(1+x)} = a$ and $\dfrac{yq}{(1+y)} = a$ so that $p = a\left(\dfrac{1+x}{x}\right)$ and $q = a\left(\dfrac{1+y}{y}\right)$. Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = a\left(\frac{1}{x} + x\right)dx + a\left(\frac{1}{y} + y\right)dy \tag{1.65}$$

By integrating (1.65), we get the required solution as

$$z = a(\log x + x) + a(\log y + y) + b \implies z = a(\log xy + x + y) + b. \tag{1.66}$$

■ **Example 1.20** Find the integral of $py + qx + pq = 0$ ■

Solution: Given equation can be written as $py + q(x+p) = 0$. Separating p and x from q and y, the given equation can be written as

$$\frac{p}{(p+x)} = -\frac{q}{y} \tag{1.67}$$

Equating each side to an arbitrary constant a, we get $\dfrac{p}{(p+x)} = a$ and $-\dfrac{q}{y} = a$ so that $p = \left(\dfrac{xa}{1-a}\right)$ and $q = -ay$. Putting the values of p and q in $dz = pdx + qdy$, we get

$$dz = \left(\frac{a}{1-a}\right)xdx - aydy \tag{1.68}$$

By integrating (1.68), we get the required solution as

$$z = \left(\frac{a}{1-a}\right)\frac{x^2}{2} - a\frac{y^2}{2} + \frac{b}{2} \implies 2z = \left(\frac{a}{1-a}\right)x^2 - ay^2 + b. \tag{1.69}$$

■ **Example 1.21** Find the integral of $z(p^2 - q^2) = x - y$ ■

Solution: Given equation can be written as $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y$. Let $\sqrt{z}dz = dZ$ so that $(2/3)z^{3/2} = Z$. Thus the given equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y \implies P^2 - Q^2 = x - y, \tag{1.70}$$

where $P = \dfrac{\partial Z}{\partial x}$ and $Q = \dfrac{\partial Z}{\partial y}$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y \tag{1.71}$$

Equating each side to an arbitrary constant a, we get $P^2 - x = a$ and $Q^2 - y = a$ so that $P = \sqrt{a+x}$ and $Q = \sqrt{a+y}$. Putting the values of P and Q in $Pdx + Qdy$, we get

$$dZ = \sqrt{a+x}dx + \sqrt{a+y}dy \tag{1.72}$$

By integrating (1.72), we get the required solution as

$$Z = (2/3)(x+a)^{3/2} + (2/3)(y+a)^{3/2} + (2/3)b$$

$$(2/3)(z)^{3/2} = (2/3)(x+a)^{3/2} + (2/3)(y+a)^{3/2} + (2/3)b$$

$$(z)^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + b \tag{1.73}$$

Exercise

Solve the following PDE:

(1) $yp = 2yx + \log q$ **Ans.** $z = (ax + x^2) + (1/a)e^{ay} + b$
(2) $p + q - 2px - 2qy + 1 = 0$ **Ans.** $z = -(a/2)\log(1-2x) + (1/2)(a+1)\log(2y+1) + b$
(3) $p^{1/3} - q^{1/3} = 3x - 3y$ **Ans.** $3x^3 - 3ax^2 + a^2x + 2y^4 - 4ay^3 + 3a^2y^2 - a^3y + b$
(4) $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$ **Ans.** $z = (1/3)(x^2+a^2)^{3/2} + (y^2-a^2)^{1/2} + b$
(5) $p^2 + q^2 = (x^2+y^2)z$ **Hint:** $\left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$
Ans. $4(z)^{1/2} = x(x^2+a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2-a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$

Special Method to solve non-linear first order partial differential equations

Standard Form-IV:(Clairaut's Form) Let the given equation is Clairaut's Form i.e. $z = px + qy + f(pq)$.
Step-I: Put the value $p = a$ and $q = b$ in given equation.
Step-II: The required solution will be $z = ax + by + f(ab)$

■ **Example 1.22** Find the integral of $z = px + qy + pq$ ■

Solution: The given equation is Clairaut's form $z = px + qy + f(pq)$. Hence the required solution can be found by putting $p = a$ and $q = b$ in given equation i.e. the solution is

$$z = ax + by + ab. \tag{1.74}$$

■ **Example 1.23** Find the integral of $(px + qy - z)^2 = 1 + p^2 + q^2$ ■

Solution: The given equation can be written as is $z = px + qx \pm \sqrt{1+p^2+q^2}$ which is Clairaut's form $z = px + qy + f(pq)$. Hence the required solution can be found by putting $p = a$ and $q = b$ in given equation i.e. the solution is

$$z = ax + bx \pm \sqrt{1+a^2+b^2}. \tag{1.75}$$

Exercise

Solve the following PDE:

(1) $(p+q)(z-px-qy)=1$ **Ans.** $z=ax+by+\dfrac{1}{a+b}$

(2) $pqz=p^2(xq+p^2)+q^2(yp+q^2)$ **Ans.** $z=ax+by+\dfrac{a^4+b^4}{ab}$

(3) $2q(z-px-qy)=1+q^2$ **Ans.** $z=ax+by+\dfrac{1+b^2}{2b}$

(4) $2\log(z-px-qy)=1+pq$ **Ans.** $z=ax+by+e^{\frac{1+ab}{2}}$.

1.2 PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

1.2.1 SOLUTION TO HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS SECOND AND HIGHER ORDER

Definition 1.2.1 — (Linear Homogeneous Partial Differential Equation of Order n). An equation of the type

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+,\ldots,a_n\frac{\partial^n z}{\partial y^n}=\phi(x,y), \tag{1.76}$$

where $a_0,a_1,\ldots,a_n$ are constants and $\phi(x,y)$ is any function of x and y, is called a homogeneous linear partial differential equation of order n with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Notations: We use the following notations:

$$\frac{\partial}{\partial x}=D \text{ and } \frac{\partial}{\partial y}=D'$$

Then equation (1.76) can be written as

$$a_0D^nz+a_1D^{n-1}D'z+a_2D^{n-2}D'^2z+,\ldots,+a_nD'^nz=\phi(x,y)$$

or

$$(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+,\ldots,+a_nD'^n)z=\phi(x,y)$$

or

$$F(D,D')z=\phi(x,y),$$

where $F(D,D')=(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+,\ldots,+a_nD'^n)$.

Working Rule to find Complementary Functions:

Step-I: Put the given equation in the standard form

$$(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+,\ldots,+a_nD'^n)z=\phi(x,y) \tag{1.77}$$

Step-II: Replacing D by m and D' by 1 in the equation (1.77), we obtain auxiliary equation (A.E.) as

$$a_0m^n+a_1m^{n-1}+a_2m^{n-2}+,\ldots,+a_n=0 \tag{1.78}$$

Step-III: Solve equation (1.78) for m.Then following cases will be arises:
Case-1: Let $m=m_1,m_2,...,m_n$ are different roots, then complementary function (C.F.) will be

$$C.F.=f_1(y+m_1x)+f_2(y+m_2x)+\ldots+f_n(y+m_nx),$$

where $f_1,f_2,...,f_n$ are arbitrary functions.
Case-2: Let r roots $m=m_1=m_2=\ldots=m_r,(r\le n)$ are equal, then complementary function (C.F.) will be

$$C.F. = f_1(y+mx) + xf_2(y+mx) + x^2 f_3(y+mx) + \dots + x^{r-1} f_r(y+x).$$

Case-3: Corresponding to a non-repeated factor D, the C.F. is taken as $f_1(y)$.
Case-4: Corresponding to a repeated factor D^r, the C.F. is taken as

$$f_1(y) + xf_2(y) + x^2 f_3(y) + \dots + x^{r-1} f_r(y).$$

Case-5: Corresponding to a non-repeated factor D', the C.F. is taken as $f_1(x)$.
Case-6: Corresponding to a repeated factor D'^r, the C.F. is taken as

$$f_1(x) + yf_2(x) + y^2 f_3(x) + \dots + y^{r-1} f_r(x).$$

Notations: We use the following notations

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}.$$

■ **Example 1.24** Solve $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = 0$ ■

Solution: The given partial differential equation can be written as

$$(D^3 - 7DD'^2 + 6D'^3)z = 0.$$

By replacing D by m and D' by 1, the auxiliary equation is

$$m^3 - 7m + 6 = 0 \implies (m-1)(m-2)(m+3) = 0.$$

Hence the roots are $m = 1, 2, -3$, which are different. Therefore general solution will be

$$z = f_1(y+x) + f_2(y+2x) + f_3(y-3x),$$

where f_1, f_2, f_3 are arbitrary functions.

■ **Example 1.25** Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ ■

Solution: By replacing D by m and D' by 1, the auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0 \implies (m-1)(m-2)(m-3) = 0.$$

Hence the roots are $m = 1, 2, 3$, which are different. Therefore general solution will be

$$z = f_1(y+x) + f_2(y+2x) + f_3(y+3x),$$

where f_1, f_2, f_3 are arbitrary functions.

■ **Example 1.26** Solve the partial differential equation $25r - 40s + 16t = 0$ ■

Solution: Given equation can be written as

$$(25D^2 - 40DD' + 16D'^2)z = 0.$$

By replacing D by m and D' by 1, the auxiliary equation is

$$25m^2 - 40m + 16 = 0 \implies (5m-4)^2 = 0.$$

Hence the roots are $m = 4/5, 4/5$, which are repeated. Therefore general solution will be

$$z = f_1(y + \frac{4}{5}x) + x f_2(y + \frac{4}{5}x)$$

or

$$z = f_1(5y + 4x) + x f_2(5y + 4x)$$

where f_1, f_2, f_3 are arbitrary functions.

■ **Example 1.27** Solve the partial differential equation $D^2 D'^2 (D + D')z = 0$ ■

Solution: The solution corresponding to the factor D^2 is $f_1(y) + x f_2(y)$
The solution corresponding to the factor D'^2 is $f_3(x) + y f_4(x)$
The solution corresponding to the factor $(D + D')$ is $f_5(y - x)$
Hence the general solution will be

$$z = f_1(y) + x f_2(y) + f_3(x) + y f_4(x) + f_5(y - x).$$

Exercise

Solve the following PDE:

(1) $(4D^2 + 12DD' + 9D'^2)z = 0$
(2) $(D^3 - 4D^2D' + 4DD'^2)z = 0$
(3) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$
(4) $r = a^2 t$
(5) $2r + 5s + 2t = 0$

Short Method to find the Particular Integral

Short Method-I (When right hand side function is of the form $\phi(ax + by)$ i.e. $F(D, D') = \phi(ax + by)$)
Let $F(D, D') = \phi(ax + by)$ be homogeneous function of D and D' of order n. Then the particular integral is defined as

$$\frac{1}{F(D, D')}\phi(v) = \frac{1}{F(a, b)} \int\int \ldots \int \phi(v)dvdv...dv,$$

where $v = ax + by$ and $F(a, b) \neq 0$.
Exceptional Case:When $F(a, b) = 0$. Let $F(D, D') = \phi(ax + by)$ be homogeneous function of D and D' of order n. Then the particular integral is defined as

$$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

■ **Example 1.28** Solve $(D^2 + 3DD' + 2D'^2)z = x + y$ ■

Solution: The solution of the auxiliary equation is $m^2 + 3m + 2 = 0$, which gives $m = -1, -2$. therefore it's complementary function (C.F.) is

$$C.F. = f_1(y - x) + f_2(y - 2x), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned} P.I. = \frac{1}{F(D,D')}\phi(ax+by) &= \frac{1}{D^2+3DD'+2D'^2}(x+y) \\ &= \frac{1}{1^2+3.1.1+2.1^2}\int\int v dv dv \\ &= \frac{1}{6}\frac{v^3}{6} \\ P.I. &= \frac{1}{36}(x+y)^3 \end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$i.e.

$$z = f_1(y-x) + f_2(y-2x) + \frac{1}{36}(x+y)^3$$

■ **Example 1.29** Solve $(D^2+2DD'+D'^2)z = e^{(2x+3y)}$ ■

Solution: The solution of the auxiliary equation is $m^2+2m+1=0$, which gives $m=-1,-1$. therefore it's complementary function (C.F.) is

$$C.F. = f_1(y-x) + x f_2(y-x), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned} P.I. = \frac{1}{F(D,D')}\phi(ax+by) &= \frac{1}{D^2+2DD'+D'^2}e^{(2x+3y)} \\ &= \frac{1}{2^2+2.2.3+3^2}\int\int e^v dv dv \\ &= \frac{1}{25}e^v \\ P.I. &= \frac{1}{25}e^{(2x+3y)} \end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$i.e.

$$z = f_1(y-x) + x f_2(y-x) + \frac{1}{25}e^{(2x+3y)}.$$

■ **Example 1.30** Solve $r - 2s + t = \sin(2x+3y)$ ■

Solution: Given equation can be written as $(D^2-2DD'+D'^2)z = \sin(2x+3y)$. Therefore the auxiliary equation is $m^2-2m+1=0$, which gives $m=1,1$. therefore it's complementary function (C.F.) is

$$C.F. = f_1(y+x) + x f_2(y+x), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned} P.I. = \frac{1}{F(D,D')}\phi(ax+by) &= \frac{1}{D^2-2DD'+D'^2}\sin(2x+3y) \\ &= \frac{1}{2^2-2.2.3+3^2}\int\int \sin(v) dv dv \\ &= \frac{1}{1}(-\sin v) \\ P.I. &= -\sin(2x+3y) \end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$i.e.

$$z = f_1(y+x) + xf_2(y+x) - \sin(2x+3y).$$

Example 1.31 Solve $4r - 4s + t = 16\log(x+2y)$ ■

Solution: Given equation can be written as $(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$. Therefore the auxiliary equation is $4m^2 - 4m + 1 = 0 \implies (2m-1)^2 = 0$, which gives $m = 1/2, 1/2$. therefore it's complementary function (C.F.) is

$$C.F. = f_1(y + \frac{1}{2}x) + xf_2(y + \frac{1}{2}x) \implies = f_1(2y+x) + xf_2(2y+x),$$

where f_1, f_2 are arbitrary function.
Now, Particular Integral (P.I.) will be

$$\begin{aligned} P.I. = \frac{1}{F(D,D')}\phi(ax+by) &= \frac{1}{4D^2 - 4DD' + D'^2} 16\log(x+2y) \\ &= 16\frac{1}{(2D-D')^2}\log(x+2y) \\ &= 16\frac{x^2}{2^2 2!}\log(x+2y) \\ P.I. &= 2x^2\log(x+2y) \end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$i.e.

$$z = f_1(2y+x) + xf_2(2y+x) + 2x^2\log(x+2y).$$

Exercise

Solve the following PDE:

(1) $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$ **Ans.** $z = f_1(y-x) + xf_2(y-2x) + 1/240(2x+3y)^3$

(2) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^(5x+6y)$ **Ans.** $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x) - (1/91)e^(5x+6y)$

(3) $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$ **Ans.** $z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - x^2\cos(2x+y)$

(4) $(D^3 - 3DD'^2 + 2D'^3)z = \sqrt{(x-2y)}$ **Ans.** $z = f_1(y+x) + xf_2(y+x) + f_3(y+2x) - \frac{8}{2835}(x-2y)^{7/2}$

(5) $(D-3D')^2(D+3D')z = e^{3x+y}$ **Ans.** $z = f_1(y+3x) + xf_2(y+3x) + f_3(y-3x) + \frac{x^2}{12}e^{3x+y}$.

Short Method to find the Particular Integral

Short Method-II (When right hand side function is of the form $\phi(x^m y^n)$ i.e. $F(D,D') = \phi(x^m y^n)$), where m and n are either integer or rational number.
Let $F(D,D') = \phi(x^m y^n)$ be homogeneous function of D and D' of order n. Then the particular integral is defined as

$$\frac{1}{F(D,D')}\phi(x^m y^n),$$

Then particular integral evaluated by expanding the symbolic function $\frac{1}{F(D,D')}$ in an infinite series of ascending power of D or D'.

Remark-1: If $n \leq m$, then $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D'}{D}$ whereas If $m \leq n$, then $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D}{D'}$.

Remark-2: Binomial expansion $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$

■ **Example 1.32** Solve $(D^2 - a^2 D'^2)z = x$. ■

Solution: The auxiliary equation is $m^2 - a^2 = 0$, which gives $m = -a, +a$. Therefore it's complementary function (C.F.) is

$$C.F. = f_1(y-ax)+f_2(y+ax), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned}
P.I. = \frac{1}{F(D,D')}\phi(x^m y^n) &= \frac{1}{D^2-a^2D'^2}(x) \\
&= \frac{1}{D^2\left[1-\left(\frac{a^2D'^2}{D^2}\right)\right]}(x) \\
&= \frac{1}{D^2}\left[1-\left(\frac{a^2D'^2}{D^2}\right)\right]^{-1}(x) \\
&= \frac{1}{D^2}\left[1+\left(\frac{a^2D'^2}{D^2}\right)+\left(\frac{a^2D'^2}{D^2}\right)^2+\ldots+\right](x) \\
&= \frac{1}{D^2}\left[1+\left(\frac{a^2D'^2}{D^2}\right)+\left(\frac{a^4D'^4}{D^4}\right)+\ldots+\right](x) \\
&= \frac{1}{D^2}\left[x+\left(\frac{a^2D'^2}{D^2}\right)x+\left(\frac{a^4D'^4}{D^4}\right)x+\ldots+\right] \\
&= \frac{1}{D^2}\left[x+\left(\frac{a^2}{D^2}\right)(D'^2x)+\left(\frac{a^4}{D^4}\right)(D'^4x)+\ldots+\right] \\
&= \frac{1}{D^2}(x) \\
P.I. &= \frac{x^3}{6}.
\end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$ i.e.

$$z = f_1(y-ax)+f_2(y+ax)+\frac{x^3}{6}.$$

■ **Example 1.33** Solve $(D^3 - D'^3)z = x^3y^3$. ■

Solution: The auxiliary equation is $m^3 - 1 = 0$, which gives $m = 1, \omega, \omega^2$, where ω and ω^2 are cube root of unity. Therefore it's complementary function (C.F.) is

$$C.F. = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x), \text{ where } f_1, f_2, f_3 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned}
P.I. = \frac{1}{F(D,D')}\phi(x^m y^n) &= \frac{1}{D^3 - D'^3}(x^3y^3) \\
&= \frac{1}{D^3\left[1-\left(\frac{D'^3}{D^3}\right)\right]}(x^3y^3) \\
&= \frac{1}{D^3}\left[1-\left(\frac{D'^3}{D^3}\right)\right]^{-1}(x^3y^3) \\
&= \frac{1}{D^3}\left[1+\left(\frac{D'^3}{D^3}\right)+\left(\frac{D'^3}{D^3}\right)^2+\left(\frac{D'^3}{D^3}\right)^3+\ldots+\right](x^3y^3) \\
&= \frac{1}{D^3}\left[1+\left(\frac{D'^3}{D^3}\right)+\left(\frac{D'^6}{D^6}\right)+\left(\frac{D'^9}{D^9}\right)+\ldots+\right](x^3y^3) \\
&= \frac{1}{D^3}\left[(x^3y^3)+\left(\frac{D'^3}{D^3}\right)(x^3y^3)+\left(\frac{D'^6}{D^6}\right)(x^3y^3)+\ldots+\right] \\
&= \frac{1}{D^3}\left[(x^3y^3)+\left(\frac{1}{D^3}\right)(D'^3(x^3y^3))+\left(\frac{1}{D^6}\right)(D'^6(x^3y^3))+\ldots+\right] \\
&= \frac{1}{D^3}\left[(x^3y^3)+\left(\frac{1}{D^3}\right)(x^3D'^3(y^3))+\left(\frac{1}{D^6}\right)(x^3D'^6(y^3))+\ldots+\right] \\
&= \frac{1}{D^3}\left[(x^3y^3)+\left(\frac{1}{D^3}\right)(x^3(3.2.1))+\left(\frac{1}{D^6}\right)(x^3(0))+\ldots+\right] \\
&= \frac{1}{D^3}\left[x^3y^3+6\left(\frac{1}{D^3}\right)(x^3)\right] \\
&= \frac{1}{D^3}(x^3y^3)+6\left(\frac{1}{D^6}\right)(x^3) \\
&= y^3\frac{1}{D^3}(x^3)+6\left(\frac{1}{D^6}\right)(x^3) \\
P.I. &= y^3\frac{x^6}{120}+\frac{x^9}{10080}.
\end{aligned}$$

Therefore the required general solution is $z = C.F. + P.I.$ i.e.

$$z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}.$$

Exercise

Solve the following PDE:

(1) $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ **Ans.** $z = f_1(y+3x) + xf_2(y+3x) + 10x^4 + 6x^3y$

(2) $(D^2-2DD'+D'^2)z = e^{x+2y}+x^3$ **Ans.** $z = f_1(y+x)+xf_2(y+x)+e^{(}x+2y)+\dfrac{x^5}{20}$

(3) $(D^3-7DD'^2-6D'^3)z = x^2+xy^2+y^3$

Short Method to find the Particular Integral

Short Method-III (General Method).

Let $F(D,D') = \phi(x,y)$ be homogeneous function of D and D' of order n. The particular integral is defined as

$$\frac{1}{F(D,D')}\phi(x,y),$$

Let the particular integral can be written as

$$\frac{1}{(D-m_1D')(D-m_2D')(D-m_3D')\ldots(D-m_nD')}\phi(x,y),$$

The we use one of the following formula

$$\frac{1}{(D-m_1D')}\phi(x,y) = \int \phi(x,c-mx)dx, \qquad \text{where } c = y+mx.$$

or

$$\frac{1}{(D+m_1D')}\phi(x,y) = \int \phi(x,c-mx)dx, \qquad \text{where } c = y-mx.$$

■ **Example 1.34** Solve $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y} = \sin x$. ■

Solution: The auxiliary equation is $m+1=0$, which gives $m=-1$. Therefore it's complementary function (C.F.) is

$$C.F. = f_1(y-x), \text{ where } f_1 \text{ is arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$\begin{aligned} P.I. = \frac{1}{F(D,D')}\phi(x,y) &= \frac{1}{(D+D')}\sin x \\ &= \int \{\sin x\}\,dx, \\ P.I. &= -\cos x \end{aligned}$$

Therefore the required general solution is $z = C.F.+P.I.$i.e.

$$z = f_1(y-x)-\cos x$$

■ **Example 1.35** Solve $(D^2-DD'-2D'^2)z = (y-1)e^x$. ■

Solution: The auxiliary equation is $m^2-m-2=0$, which gives $m=-1,2$. Therefore it's complementary function (C.F.) is

$$C.F. = f_1(y-x)+f_2(y+2x), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$
\begin{aligned}
P.I. = \frac{1}{F(D,D')}\phi(x,y) &= \frac{1}{(D+D')(D-2D')}(y-1)e^x \\
&= \frac{1}{(D+D')}\left\{\frac{1}{(D-2D')}(y-1)e^x\right\} \\
&= \frac{1}{(D+D')}\int\{(c-2x-1)e^x\}\,dx, \\
&\qquad \therefore \quad c = y+2x \\
&= \frac{1}{(D+D')}\left\{(c-2x-1)e^x - \int(-2)e^x dx\right\} \\
&= \frac{1}{(D+D')}\{(c-2x-1)e^x + 2e^x\} \\
&= \frac{1}{(D+D')}\{(c-2x+1)e^x\} \\
&= \frac{1}{(D+D')}\{(y+2x-2x+1)e^x\} \\
&\qquad \therefore c = y+2x \\
&= \frac{1}{(D+D')}\{(y+1)e^x\} \\
&= \int(c'+x+1)e^x dx \\
&\qquad \therefore c' = y-x \\
P.I. &= (c'+x+1)e^x - \int(1.e^x)dx = (c'+x+1)e^x - e^x = ye^x \\
&\qquad \therefore c' = y-x.
\end{aligned}
$$

Therefore the required general solution is $z = C.F. + P.I.$ i.e.

$$z = f_1(y-x) + f_2(y+2x) + ye^x.$$

■ **Example 1.36** Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$. ■

Solution: The auxiliary equation is $m^2 - m - 2 = 0$, which gives $m = -1, 2$. Therefore it's complementary function (C.F.) is

$$C.F. = f_1(y-x) + f_2(y+2x), \text{ where } f_1, f_2 \text{ are arbitrary function.}$$

Now, Particular Integral (P.I.) will be

$$
\begin{aligned}
P.I. = \frac{1}{F(D,D')}\phi(x,y) &= \frac{1}{(D-2D')(D+D')}\{(2x^2+xy-y^2)\sin xy - \cos xy\} \\
&= \frac{1}{(D-2D')(D+D')}\{(2x-y)(x+y)\sin xy - \cos xy\} \\
&= \frac{1}{(D-2D')}\int \{(2x-x-c)(x+x+c)\sin x(x+c) - \cos x(x+c)\}\,dx, \\
&\qquad \therefore \quad c = y-x \\
&= \frac{1}{(D-2D')}\int \left\{(x-c)(2x+c)\sin(x^2+cx) - \cos(x^2+cx)\right\}dx \\
&= \frac{1}{(D-2D')}\left\{-(x-c)\cos(x^2+cx) + \int \cos(x^2+cx)dx - \int \cos(x^2+cx)dx\right\} \\
&= \frac{1}{(D-2D')}\{(y-2x)\cos xy\} \\
&\qquad \therefore c = y-x \\
&= \int (c'-2x-2x)\cos x(c'-2x)dx \\
&= \\
&\qquad \therefore c' = y+2x \\
&= \int (c'-4x)\cos(xc'-2x^2)dx \\
&= \quad Let \quad xc'-2x^2 = t \quad so \quad that \quad (c'-4x)dx = dt \\
P.I. &= \sin(c'x-2x^2) = \sin xy.
\end{aligned}
$$

Therefore the required general solution is $z = C.F. + P.I.$ i.e.

$$z = f_1(y-x) + f_2(y+2x) + \sin xy.$$

Exercise

Solve the following PDE:

(1) $(D^2-4D'^2)z = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$ **Ans.** $z = f_1(y+2x) + f_2(y-2x) + x\log y + y\log x + 3x$

(2) $r+s-6t = y\sin x$ **Ans.** $z = f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x$

(3) $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$ **Ans.** $z = f_1(y-x) + xf_2(y-x) + x\sin y$

(4) $r-t = \tan^3 x\tan y - \tan x\tan^3 y$ [AKU2019] **Ans.** $z = f_1(y-x) + xf_2(y+x) + 1/2\tan y\tan x.$

Non-Homogeneous Linear Partial Differential Equations with Constants Coefficients

Definition 1.2.2 A linear partial differential equations with constant coefficients, which are not homogeneous are called Non-homogeneous.

■ **Example 1.37** $2\frac{\partial^3 z}{\partial z^3} - 3\frac{\partial^2 z}{\partial z^2} + \frac{\partial z}{\partial z} + 2z = x + 2y$ ■

■ **Example 1.38** $\frac{\partial^3 z}{\partial z^3} + \frac{\partial z}{\partial z} - 4z = \sin(x+2y)$ ■

Definition 1.2.3 A linear differential operator $F(D,D')$ is known as **reducible**, if it can be written as the product of linear factors of the form $aD + bD' + c$, where a, b and c are constants. $F(D,D')$ is known as **irreducible**, if it is not reducible.

■ **Example 1.39** $D^2 - D'^2$ is reducible because it can be written as a linear factor $(D^2 - D'^2) = (D-D')(D+D')$ ■

■ **Example 1.40** $D^3D' - DD'^3$ is reducible because it can be written as a linear factor $D^3D' - DD'^3 = DD'(D-D')(D+D')$ ■

■ **Example 1.41** $D^2 - D'^3$ is irreducible because it can not be written as a linear factor. ■

Working rule for finding C.F. of reducible non-homogeneous linear partial differential equations with constants coefficients.

Let the given reducible non-homogeneous linear partial differential equations with constants coefficients be $F(D,D')z = \phi(x,y)$
Step-I: Factorize $F(D,D')$ into linear factors.
Step-II: Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of complementary function is taken as $e^{(cx/b)} f_1(by+ax)$, if $b \neq 0$.
Step-III: Corresponding to repeated factor $(bD - aD' - c)^r$, the part of complementary function is taken as $e^{(cx/b)}\left[f_1(by+ax) + xf_2(by+ax) + x^2 f_3(by+ax) + \ldots + x^{(r-1)} f_r(by+ax)\right]$, if $b \neq 0$.
Step-IV: Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of complementary function is taken as $e^{-(cy/a)} f_{-1}(by+ax)$, if $a \neq 0$.
Step-V: Corresponding to repeated factor $(bD - aD' - c)^r$, the part of complementary function is taken as $e^{-(cy/a)}\left[f_1(by+ax) + yf_2(by+ax) + y^2 f_3(by+ax) + \ldots + y^{(r-1)} f_r(by+ax)\right]$, if $a \neq 0$, $f_1, f_2, f_3, \ldots, f_r$ are arbitrary functions.

■ **Example 1.42** Solve the PDE $(D^2 - D'^2 + D - D')z = 0$. ■

Solution: The given PDE $(D^2 - D'^2 + D - D')z = 0$ is reducible because it can be written as a linear factor

$$[(D-D')(D+D') + D - D']z = 0 \implies (D-D')(D+D'+1)z = 0.$$

By comparing $(D-D')$ with $(bD - aD' - c)$, we get $b = 1$, $a = 1$ and $c = 0$. Now part of complementary function (C.F.) corresponding to the factor $(D-D')$ is

$$e^{(0.x/(1))} f_1(1.y + 1.x) \implies f_1(y+x).$$

Again by comparing $(D+D'+1)$ with $(bD - aD' - c)$, we get $b = 1$, $a = -1$ and $c = -1$. Now part of complementary function (C.F.) corresponding to the factor $(D+D'+1)$ is

$$e^{((-1).x/(1))}\phi(1.y+(-1).x) \implies e^{-x}f_2(y-x).$$

Hence the required solution is

$$z = f_1(y+x)+e^{-x}f_2(y-x),$$

where f_1 and f_2 are arbitrary function.

■ **Example 1.43** Solve the PDE $r+2s+t+2p+2q+z=0$. ■

Solution: The given PDE can be written as $(D^2+2DD'+D'2+2D+2D'+1)z=0$, which is reducible because it can be written as a linear factor

$$[(D+D')^2+2D+2D'+1]z=0 \implies [(D+D')^2+2(D+D')+1]z=0.$$

$$[(D+D'+1)^2]z=0.$$

By comparing $(D+D'+1)$ with $(bD-aD'-c)$, we get $b=1$, $a=-1$ and $c=-1$. Now part of complementary function (C.F.) corresponding to the factor $(D+D'+1)^2$ is

$$e^{((-1).x/(1))}\{f_1(1.y-1.x)+xf_2(1.y-1.x)\} \implies e^{-x}\{f_1(y-x)+xf_2(y-x)\}.$$

Hence the required solution is

$$z = e^{-x}\{f_1(y-x)+xf_2(y-x)\},$$

where f_1 and f_2 are arbitrary function.

■ **Example 1.44** Solve the PDE $(3D-5)(7D'+2)DD'(2D+3D'+5)z=0$. ■

Solution: The given PDE is in a linear factor. Hence the required solution is

$$z = e^{5x/3}f_1(3y)+e^{-(2y/7)}f_2(7x)+f_3(y)+f_4(x)+e^{(-5x/2)}f_5(2y-3x),$$

where f_1, f_2, f_3, f_4 and f_5 are arbitrary function.

Exercise

Solve the following PDE:

(1) $(D^2-DD'+D'-1)z=0$ **Ans.** $z=e^x f_1(y)+e^{-x}f_2(y+x)$
(2) $s+p-q-z=0$ **Ans.** $e^x f_1(y)+e^{-y}f_2(x)$
(3) $(D^2-DD'-2D'^2+2D+2D')z=0$ **Ans.** $z=f_1(y-x)+e^{-2x}f_2(y+2x)$
(4) $(D^2-D'^2+D-D')z=0$ **Ans.** $z=f_1(y+x)+e^{-x}f_2(y-x)$
(5) $(DD'+aD+bD'+ab)z=0$ **Ans.** $z=e^{-bx}f_1(y)+e^{-ay}f_2(x)$

Working rule for finding C.F. of irreducible non-homogeneous linear partial differential equations with constants coefficients.

Let the given irreducible non-homogeneous linear partial differential equations with constants coefficients be $F(D,D')z=\phi(x,y)$
Step-I: If necessary Factorize $F(D,D')$ in the form $F_1(D,D')F_2(D,D')$, where $F_1(D,D')$ consists of product of linear factors in D,D' and $F_2(D,D')$ consists of product of irreducible factors in D,D'.
Step-II: Write the part of C.F. of linear factors $F_1(D,D')$ as usual method
Step-III: Write the part of C.F. of irreducible factors $F_2(D,D')$ by taking a trial solution

$$C.F. = \sum Ae^{hx+ky},$$

where A, h and k are arbitrary constants such that $F(h,k) = 0$

Step-IV: Adding the part of C.F. of reducible factors $F_1(D,D')$, obtained in Step-II and part of C.F. of irreducible factors $F_2(D,D')$, obtained in Step-III.

■ **Example 1.45** Solve the PDE $(D - D'^2)z = 0$. ■

Solution: Here $D - D'^2$ is not a linear factors in D and D'. Let the trial solution of given equation is

$$z = \sum Ae^{hx+ky}$$

Then $Dz = Ahe^{hx+ky}$ and $D'^2z = Ak^2e^{hx+ky}$. Putting these values in the given equation, we get

$$Ahe^{hx+ky} - Ak^2e^{hx+ky} = 0 \implies A(h-k^2)e^{hx+ky} = 0$$

$$h - k^2 = 0 \implies h = k^2.$$

Replacing h by k^2, the most general solution of the given equation is

$$z = \sum Ae^{k^2x+ky},$$

where A and k are arbitrary constant.

■ **Example 1.46** Solve the PDE $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$. ■

Solution: Here $(D - 2D' - 1)$ is a linear factors in D and D'. Therefore its complementary function (C.F.) is $e^x f_1(y+2x)$, where f_1 is an arbitrary function. To find the complementary function (C.F.) corresponding factor $(D - 2D'^2 - 1)z$. Let the trial solution of this factor is

$$z = \sum Ae^{hx+ky}$$

Then $Dz = Ahe^{hx+ky}$ and $D'^2z = Ak^2e^{hx+ky}$. Putting these values in the factor $(D - 2D'^2 - 1)z$, we get

$$Ahe^{hx+ky} - 2Ak^2e^{hx+ky} - \sum Ae^{hx+ky} = 0 \implies A(h - 2k^2 - 1)e^{hx+ky} = 0$$

$$h - 2k^2 - 1 = 0 \implies h = 2k^2 + 1.$$

Replacing h by $2k^2 + 1$, the complementary function (C.F.) corresponding factor $(D - 2D'^2 - 1)z$ is $C.F. = \sum Ae^{(k^2+1)x+ky}$. Now the required general solution of the given equation is

$$z = e^x f_1(y+2x) + \sum Ae^{(k^2+1)x+ky},$$

where A and k are arbitrary constant.

■ **Example 1.47** Solve the PDE $(2D^4 - 3D^2D' + D'^2)z = 0$. ■

Solution: Given equation can be written as $(2D^2 - D')(D^2 - D')z = 0$. To find the complementary function (C.F.) corresponding factor $(D^2 - D')z$. Let the trial solution of this factor is

$$z = \sum Ae^{hx+ky}$$

Then $D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Putting these values in the factor $(D^2 - D')z$, we get

$$Ah^2e^{hx+ky} - Ake^{hx+ky} = 0 \implies A(h^2-k)e^{hx+ky} = 0$$

$$h^2 - k = 0 \implies k = h^2.$$

Replacing k by h^2, the complementary function (C.F.) corresponding factor $(D^2 - D')z$ is $C.F. = \sum Ae^{hx+h^2y}$.
Again to find the complementary function (C.F.) corresponding factor $(2D^2 - D')z$. Let the trial solution of this factor is

$$z = \sum A_1e^{h_1x+k_1y}$$

Then $D^2z = A_1h_1^2e^{h_1x+k_1y}$ and $D'z = A_1k_1e^{h_1x+k_1y}$. Putting these values in the factor $(2D^2 - D')z$, we get

$$2A_1h_1^2e^{h_1x+k_1y} - A_1k_1e^{h_1x+k_1y} = 0 \implies A_1(2h_1^2 - k_1)e^{h_1x+k_1y} = 0$$

$$2h_1^2 - k_1 = 0 \implies k_1 = 2h_1^2.$$

Replacing k_1 by $2h_1^2$, the complementary function (C.F.) corresponding factor $(2D^2 - D')z$ is $C.F. = \sum A_1e^{h_1x+2h_1^2y}$. Now the required general solution of the given equation is

$$z = \sum Ae^{hx+h^2y} + \sum A_1e^{h_1x+2h_1^2y},$$

where A, A_1, h and h_1 are arbitrary constant.

Exercise

Solve the following PDE:

(1) $(D^2 + D'^2)z = n^2z$ **Ans.** $z = \sum Ae^{n(x\cos\alpha + y\sin\alpha)}$ (Here $h = n\cos\alpha$ and $k = n\sin\alpha$)
(2) $(D + 2D' - 3)(D^2 + D')z = 0$ **Ans.** $z = e^3xf_1(y-2x) + \sum Ae^{hx-h^2y}$
(3) $(D^2 - D')z = 0$ **Ans.** $z = \sum Ae^{hx+h^2y}$
(4) $(2D^2 - D'^2 + D)z = 0$ **Ans.** $z = \sum Ae^{hx+ky}$, where $2h^2 - k^2 + h = 0$.

Working rule for finding Particular Integral P.I. of reducible/irreducible non-homogeneous linear partial differential equations with constants coefficients.

Let the given reducible/irreducible non-homogeneous linear partial differential equations with constants coefficients be $F(D,D')z = \phi(x,y)$
Case-I: When $\phi(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$.
Then, we get the P.I. by replacing D by a and D' by b. i.e.

$$P.I. = \frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}$$

■ **Example 1.48** Solve the PDE $(DD' + aD + bD' + ab)z = e^{mx+ny}$. ■

Solution: The given equation can be written as $(D+b)(D'+a)z = e^{mx+ny}$, which is reducible. Hence complementary function (C.F.) is

$$C.F. = e^{-bx} f_1(y) + e^{-ay} f_2(x), \ f_1 \text{ and } f_2 \text{ are arbitrary constant.}$$

and

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{(D+b)(D'+a)} e^{mx+ny} = \frac{1}{(m+b,n+a)} e^{mx+ny}.$$

Hence the required solution is

$$z = e^{-bx} f_1(y) + e^{-ay} f_2(x) + \frac{1}{(m+b,n+a)} e^{mx+ny}.$$

■ **Example 1.49** Solve the PDE $(D^2 - D'^2 + D - D')z = e^{2x+3y}$. ■

Solution: The given equation can be written as

$$[(D-D')(D+D') + D - D']z = e^{2x+3y} \implies (D-D')(D+D'+1)z = e^{2x+3y},$$

which is reducible. Hence it's complementary function (C.F.) is

$$C.F. = f_1(y+x) + e^{-x} f_2(y-x), \ f_1 \text{ and } f_2 \text{ are arbitrary constant.}$$

and

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{(D-D')(D+D'+1)} e^{2x+3y} = \frac{1}{(2-3)(2+3+1)} e^{2x+3y} \implies = -\frac{1}{6} e^{2x+3y}.$$

Hence the required solution is

$$z = f_1(y+x) + e^{-x} f_2(y-x) - \frac{1}{6} e^{2x+3y}.$$

■ **Example 1.50** Solve the PDE $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$. ■

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum A e^{hx+ky}.$$

Therefore we have $Dz = \sum A h e^{hx+ky}$, $D^2 z = \sum A h^2 e^{hx+ky}$ and $DD'z = \sum A hk e^{hx+ky}$. Put all these values in given equation $(D^2 - 4DD' + D - 1)z = 0$, we have

$$\sum A h^2 e^{hx+ky} - 4 \sum A hk e^{hx+ky} + \sum A h e^{hx+ky} - \sum A e^{hx+ky} = 0.$$

$$\implies \sum A(h^2 - 4hk + h - 1) e^{hx+ky} = 0 \implies (h^2 - 4hk + h - 1) = 0.$$

$$\implies k = \frac{(h^2 + h - 1)}{4h}.$$

Thus $C.F. = \sum A e^{hx+ky}$, where $k = \dfrac{(h^2+h-1)}{4h}$

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{(D^2 - 4DD' + D - 1)} e^{3x-2y} = \frac{1}{(3^2 - 4.3.(-2) + 3 - 1)} e^{3x-2y} = \frac{1}{35} e^{3x-2y}.$$

Hence the required solution is

$$z = \sum Ae^{hx+ky} + \frac{1}{35}e^{3x-2y}, \text{ where } k = \frac{(h^2+h-1)}{4h}.$$

Exercise

Solve the following PDE:

(1) $(D-D'-1)(D-D'-2)z = e^{2x-y}$ **Ans.** $z = e^x f_1(y+x) + e^{2x} f_2(y+x) + (1/2)e^{2x-1}$.

(2) $(D^3-3DD'+D+1)z = e^{2x+3y}$ **Ans.** $z = \sum Ae^{hx+ky} - \frac{1}{7}e^{2x+3y}$, where $k = \frac{(h^3+h+1)}{3h}$.

(3) $(D^2-D'^2-3D')z = e^{x+2y}$ **Ans.** $z = \sum Ae^{hx+ky} - \frac{1}{9}e^{x+2y}$, where $h = \sqrt{k^2+3k}$.

(4) $(D^2-D'^2+D+3D'-2)z = e^{x-y}$ **Ans.** $z = e^{-2x} f_1(y+x) + e^x f_2(y-x) - (1/4)e^{x-y}$.

Case-II: When $\phi(x,y) = \sin(ax+by)$ or $\cos(ax+by)$.
Then, we get the P.I., by replacing D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$ in

$$P.I. = \frac{1}{F(D,D')}\sin(ax+by) \text{ or } \frac{1}{F(D,D')}\cos(ax+by),$$

provided denominator should not be zero.

■ **Example 1.51** Solve the PDE $(D^2+DD'+D'-1)z = \sin(x+2y)$. ■

Solution: The given equation can be written as linear factors $(D+1)(D+D'-1)z = \sin(x+2y)$. Hence it's complementary function (C.F.) is

$$C.F. = e^{-x} f_1(y) + e^x f_2(y-x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\sin(x+2y) = \frac{1}{(D^2+DD'+D'-1)}\sin(x+2y) =$$
$$\frac{1}{(-1^2+(-1.2)+D'-1)}\sin(x+2y) = \frac{1}{D'-4}\sin(x+2y).$$

$$P.I. = (D'+4)\frac{1}{D'^2-4^2}\sin(x+2y) \implies (D'+4)\frac{1}{-2^2-16}\sin(x+2y).$$

$$P.I. = -\frac{1}{20}(D'+4)\sin(x+2y) \implies -\frac{1}{20}\left[D'\sin(x+2y)+4\sin(x+2y)\right].$$

$$P.I. = -\frac{1}{20}\left[2\cos(x+2y)+4\sin(x+2y)\right].$$

Hence the required solution is

$$z = e^{-x} f_1(y) + e^x f_2(y-x) - \frac{1}{10}\left[\cos(x+2y)+2\sin(x+2y)\right].$$

■ **Example 1.52** Solve the PDE $(D-D'^2)z = \cos(x-3y)$. ■

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $Dz = \sum Ahe^{hx+ky}$ and $D'^2 z = \sum Ak^2 e^{hx+ky}$. By Putting these values in given equation $(D - D'^2)z = 0$, we have

$$\sum Ahe^{hx+ky} - \sum Ak^2 e^{hx+ky} = 0 \implies \sum A(h-k^2)e^{hx+ky} = 0.$$

$$h - k^2 = 0 \implies h = k^2.$$

Hence

$$C.F. = \sum Ae^{k^2x+ky}.$$

$$P.I. = \frac{1}{F(D,D')}\cos(ax+by) = \frac{1}{(D-D'^2)}\cos(x-3y) = \frac{1}{(D-(-3^2))}\cos(x-3y) = \frac{1}{D+9}\cos(x-3y).$$

$$P.I. = (D-9)\frac{1}{D^2-9^2}\cos(x-3y) \implies (D-9)\frac{1}{-1^2-81}\cos(x-3y).$$

$$P.I. = -\frac{1}{82}(D-9)\cos(x-3y) \implies -\frac{1}{82}\left[D\cos(x-3y) - 9\cos(x-3y)\right].$$

$$P.I. = -\frac{1}{82}\left[-\sin(x-3y) - 9\cos(x-3y)\right] \implies = \frac{1}{82}\left[\sin(x-3y) + 9\cos(x-3y)\right].$$

Hence the required solution is

$$z = \sum Ae^{k^2x+ky} + \frac{1}{82}[\sin(x-3y) + 9\cos(x-3y)].$$

Exercise

Solve the following PDE:

(1) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = \cos(x+2y)$ **Ans.** $z = e^x f_1(y) + e^{-x} f_2(y+x) + (1/2)\sin(x+2y)$.

(2) $(D^2 - DD' - 2D)z = \sin(3x+4y)$ **Ans.** $z = f_1(y) + e^{2x} f_2(y+x) + (1/15)\,[\sin(3x+4y) + 2\cos(3x+4y)]$.

(3) $(D-D'-1)(D-D'-2)z = \sin(2x+3y)$ **Ans.** $z = e^x f_1(y+x) + e^{2x} f_2(y+x) + (1/10)\,[\sin(2x+3y) - 3\cos(2x$

(4) $(D^2 - D')z = \cos(3x-y)$ **Ans.** $z = \sum Ae^{hx+h^2y} - \dfrac{1}{82}\,[-\sin(3x-y) + 9\cos(3x-y)]$.

Case-III: When $\phi(x,y) = x^m y^n$.
Then

$$P.I. = \frac{1}{F(D,D')}x^m y^n = [F(D,D')]^{-1} = x^m y^n.$$

which is evaluated by expanding $[F(D,D')]^{-1}$ in ascending powers of D/D' or D'/D as the case may be.

■ **Example 1.53** Solve the PDE $s + p - q = z + xy$. ■

The given equation can be rewritten as $(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$. The complementary function (C.F.) is

$$e^x f_1(y) + e^{-y} f_2(x), \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

$$= -\frac{1}{(1-D)(1+D')} xy \implies -(1-D)^{-1}(1+D')^{-1} xy.$$

$$= -\left[1 + D + D^2 + \ldots\right]\left[1 - D' + D'^2 - \ldots\right] xy.$$

$$= -\left[1 + D + D^2 + \ldots\right]\left[xy - D'(xy) + D'^2(xy) - \ldots\right].$$

$$= -\left[1 + D + D^2 + \ldots\right](xy - x).$$

$$= -\left[(xy - x) + D(xy - x) + D^2(xy - x) + \ldots\right].$$

$$= -\left[(xy - x) + (y - 1)\right].$$

$$= -xy + x - y + 1.$$

Therefore the required solution is

$$z = e^x f_1(y) + e^{-y} f_2(x) - xy + x - y + 1.$$

■ **Example 1.54** Solve the PDE $r - s + p = 1$. ■

The given equation can be rewritten as $(D^2 - DD' + D)z = 1 \implies D(D - D' + 1)z = 1$. The complementary function (C.F.) is

$$f_1(y) + e^{-x} f_2(y + x), \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{D(D - D' + 1)}.1$$

$$= \frac{1}{D}(1 + D - D')^{-1}.1 \implies \frac{1}{D}\left[1 - (D - D') + (D - D')^2 - \ldots\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x + y) + x.$$

■ **Example 1.55** Solve the PDE $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$ ■

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$$f_1(y)+e^x f_2(y-x)+e^{2x} f_3(y-3x), \text{ where } f_1, f_2 \text{ and } f_3 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{D(D+D'-1)(D+3D'-2)}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}(1-(D+D')^{-1})\left\{1-\frac{D+3D'}{2}\right\}^{-1}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\dots\right\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\dots+\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\frac{(D+D')(D+3D')}{2}\dots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\frac{(D+D')(D+3D')}{2}\dots\right\}(x^2-4xy+2y^2).$$

$$= \frac{1}{2D}\left\{1+\frac{3D}{2}+\frac{5D'}{2}+\frac{7D^2}{4}+\frac{19D'^2}{4}+\frac{11DD'}{2}\dots\right\}(x^2-4xy+2y^2).$$

$$= \frac{1}{2D}\left\{(x^2-4xy+2y^2)+3(x-2y)+5(2y-2x)+\frac{7}{2}+19-22\right\}.$$

$$= \frac{1}{2D}\left\{x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4xy+\frac{x}{2}\right\}.$$

Therefore the required solution is

$$z = f_1(y)+e^x f_2(y-x)+e^{2x} f_3(y-3x)+\frac{1}{2}\left\{\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4xy+\frac{x}{2}\right\}.$$

Exercise

Solve the following PDE:

(1) $(D+D'-1)(D+2D'-3)z = 4+3x+6y$ **Ans.** $z = e^x f_1(y-x)+e^{3x} f_2(y-2x)+6+x+2y$

(2) $(D^2 - D'^2 - 3D + 3D')z = xy$ **Ans.** $z = f_1(y+x) + e^{3x} f_2(y-x) - (1/6)x^2y - (x^2/9) - (2x/27) - (x^3/18)$.

(3) $r - t + p + 3q - 2 = x^2y$ **Ans.** $z = e^{-2x} f_1(y+x) + e^x f_2(y-x) - (4x^2y + 4xy + 6x^2 + 6y + 12x + 21)/8$.

Case-IV: When $\phi(x,y) = Ve^{ax+by}$, where V is a ny function of x and y.
Then

$$P.I. = \frac{1}{F(D,D')} Ve^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V.$$

Example 1.56 Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$. ■

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum A e^{hx+ky}.$$

So that $D^2 z = \sum A h^2 e^{hx+ky}$ and $D'z = \sum A k e^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum A h^2 e^{hx+ky} - \sum A k e^{hx+ky} = 0 \implies \sum A(h^2 - k) e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum A e^{hx+h^2y}.$$

$$P.I. = \frac{1}{F(D,D')} \phi(x,y) = \frac{1}{(D^2 - D')} xe^{ax+a^2y} = e^{ax+a^2y} \frac{1}{((D+a)^2 - (D'+a^2)} x$$

$$= e^{ax+a^2y} \frac{1}{D^2 + 2aD - D'} x = e^{ax+a^2y} \frac{1}{2aD} \frac{1}{1 + \left(\frac{D^2 - D'}{2aD}\right)} x$$

$$= e^{ax+a^2y} \frac{1}{2aD} \left[1 + \left(\frac{D^2 - D'}{2aD}\right)\right]^{-1} x$$

$$= e^{ax+a^2y} \frac{1}{2aD} \left[1 - \left(\frac{D^2 - D'}{2aD}\right) + \left(\frac{D^2 - D'}{2aD}\right)^2 - \dots\right] x$$

$$= e^{ax+a^2y} \frac{1}{2aD} \left[x - \left(\frac{D}{2a} x - \frac{D'}{2aD} x\right) + \dots\right]$$

$$= e^{ax+a^2y} \frac{1}{2aD} \left[x - \frac{1}{2a}\right] \implies = e^{ax+a^2y} \frac{1}{2a} \left[\frac{x^2}{2} - \frac{1}{2a} x\right]$$

Hence the required solution is

$$z = \sum A e^{hx+h^2y} + e^{ax+a^2y} \left[\frac{x^2}{4a} - \frac{x}{4a^2}\right].$$

■ **Example 1.57** Solve the PDE $(D-3D'-2)^2z = 2e^{2x}\sin(y+3x)$ ■

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$$e^{2x}[f_1(y+3x)+xf_2(y+3x)], \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies 2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}[f_1(y+3x)+xf_2(y+3x)]+2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Exercise

Solve the following PDE:

(1) $(3D^2-2D'^2+D-1)z = 4e^{x+y}\cos(x+y)$ **Ans.** $z = \sum Ae^{hx+ky}+(4/3)e^{x+y}\sin(x+y)$, where h and k are related by $3h^2-2k^2+h-1$.
(2) $(D-3D'-2)^2z = 2e^{2x}\tan(y+3x)$ **Ans.** $z = e^{2x}f_1(y+3x)+xf_2(y+3x)+x^2e^{2x}\tan(y+3x)$.
(3) $r-3s+2t-p+2q = (2+4x)e^{-y}$ **Ans.** $z = f_1(y+2x)+e^xf_2(y+x)+x^2e^{-y}$.

Case-V: When $\phi(x,y) = e^{ax+by}$ and $F(a,b) = 0$.
Then

$$P.I. = \frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(D,D')}e^{ax+by}.1 = e^{ax+by}\frac{1}{F(D+a,D'+b)}.x^0y^0$$

■ **Example 1.58** Solve the PDE $(D^2-D')z = e^{x+y}$. ■

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2-D')z = 0$, we have

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2-k)e^{hx+ky} = 0.$$

$$h^2-k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2-D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2-(D'+1)}.1$$

$$= e^{x+y}\frac{1}{D^2+2D-D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1+\left[\left(\frac{D^2-D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1+\left(\frac{D^2-D'}{2D}\right)\right]^{-1}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D^2-D'}{2D}\right)+\left(\frac{D^2-D'}{2D}\right)^2-...\right].1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D}{2}.1-\frac{D'}{2D}.1\right)+...\right]$$

$$= e^{x+y}\frac{1}{2D}(1) \implies = e^{x+y}\left(\frac{1}{2}\right)x$$

Hence the required solution is

$$z = \sum Ae^{hx+h^2y} + \frac{x}{2}e^{x+y}.$$

Exercise

Solve the following PDE:

(1) $(D^2-D'^2-3D+3D')z = e^{x+2y}$ **Ans.** $z = f_1(y+x)+e^{3x}f_2(y-x)-xe^{x+2y}$.
(2) $(D^2-D')z = e^{2x+y}$ **Ans.** $z = \sum Ae^{hx+h^2y} - \frac{1}{3}e^{2x+y}$
(3) $r-4s+4t+p-2q = e^{x+y}$ **Ans.** $z = f_1(y+2x)+e^{-x}f_2(y+2x)-xe^{x+y}$.

Classification of second order partial differential equations

Consider a general second order partial differential equation for a function of two variables x and y in the form

$$\text{Rr+Ss+Tt+f(x,y,z,p,q)=0,} \tag{1.79}$$

where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$. Also R, S and T are continuous functions of x and y only possessing partial derivatives defined in a domain D on the $x-y$ plan. Then the given equation (1.79) is said to be

- Hyperbolic at a point (x,y) in domain D if $S^2-4RT>0$
- Parabolic at a point (x,y) in domain D if $S^2-4RT=0$
- Elliptic at a point (x,y) in domain D if $S^2-4RT<0$

■ **Example 1.59** Classify the following partial differential equation

1. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$.
2. $2\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = 2$.
3. $(xy-1)r - 2(x^2y^2-1)s - (xy+1)t + xp + yq = 0$

■

Solution (1.) The given equation can be written as $r - t = 0$. Comparing the given equation with

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0,$$

We have $R = 1$, $S = 0$ and $T = -1$. Put these values in $S^2 - 4RT = (0)^2 - 4.(1)(-1) \implies S^2 - 4RT = 4 > 0$. Therefore the given equation is hyperbolic.
Solution (2.) The given equation can be written as $2r + s + 3t - 2 = 0$. Comparing the given equation with

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0,$$

We have $R = 2$, $S = 1$ and $T = 3$. Put these values in $S^2 - 4RT = (1)^2 - 4.(2)(3) \implies S^2 - 4RT = -23 < 0$. Therefore the given equation is elliptic.
Solution (3.) Comparing the given equation with

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0,$$

We have $R = (xy-1)$, $S = -2(x^2y^2-1)$ and $T = -(xy+1)$. Put these values in

$$S^2 - 4RT = (-2(x^2y^2-1))^2 - 4.((xy-1)).(-(xy+1)) \implies 4(x^2y^2-1))^2 + 4.((x^2y^2-1)).$$

$$S^2 - 4RT = 4x^2y^2(x^2y^2-1).$$

Case-1: Either $x = 0$ or $y = 0$ or both $x = y = 0$. In this case $S^2 - 4RT = 0$, hence given equation is parabola.
Case-2: If $xy = \pm 1$, then in this case $S^2 - 4RT = 0$, hence given equation is parabola.
Case-3: If $x^2y^2 > 1$, then in this case $S^2 - 4RT > 0$, hence given equation is hyperbola.
Case-4: If $x^2y^2 < 1$, then in this case $S^2 - 4RT < 0$, hence given equation is elliptic.

■ **Example 1.60 — (AKU-CE-II,2019).** Classify the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} + t\dfrac{\partial^2 u}{\partial x \partial t} + x\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + 6u = 0$. ■

Solution: Comparing the given equation with

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0,$$

We have $R = x$, $S = t$ and $T = 1$. Put these values in

$$S^2 - 4RT = (t)^2 - 4.(x).(1) \implies S^2 - 4RT = t^2 - 4x.$$

Case-1: If $x = t^2/4$, then in this case $S^2 - 4RT = 0$, hence given equation is parabola.
Case-2: If $x < t^2/4$, then in this case $S^2 - 4RT > 0$, hence given equation is hyperbola.
Case-3: If $x > t^2/4$, then in this case $S^2 - 4RT < 0$, hence given equation is elliptic.

Example 1.61 — (AKU-CE-II,2019). The region in which the following partial differential equation $x^3\dfrac{\partial^2 u}{\partial x^2}+3\dfrac{\partial^2 u}{\partial x\partial y}+27\dfrac{\partial^2 u}{\partial y^2}+5u=0.$ ∎

Solution: Comparing the given equation with

$$Rr+Ss+Tt+f(x,y,z,p,q)=0,$$

We have $R=x^3$, $S=3$ and $T=27$. Put these values in

$$S^2-4RT=(3)^2-4.(x^3).(27) \implies S^2-4RT=9-108x^3.$$

Case-1: If $x=(1/12)^{1/3}$, then in this case $S^2-4RT=0$, hence given equation is parabola.
Case-2: If $x<(1/12)^{1/3}$, then in this case $S^2-4RT>0$, hence given equation is hyperbola.
Case-3: If $x>(1/12)^{1/3}$, then in this case $S^2-4RT<0$, hence given equation is elliptic.

Exercise

Classify the following PDE:

(1.) $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0.$
(2.) $\dfrac{\partial^2 z}{\partial x^2}+4\dfrac{\partial^2 z}{\partial x\partial y}+4\dfrac{\partial^2 z}{\partial y^2}=0.$
(3.) $xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$
(4.) $x^2(y-1)r-x(y^2-1)s+y(y-1)t+xyp-q=0$

METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.
Notations: Let $u(x,t)$ is a function of two variable x and t. We use the following notations:

$$\frac{\partial u}{\partial x}=u_x=u_x(x,t),\quad \frac{\partial u}{\partial t}=u_t=u_t(x,t),\quad \left(\frac{\partial u}{\partial x}\right)_{x=\pi}=u_x(\pi,t),\quad \left(\frac{\partial u}{\partial t}\right)_{t=0}=u_t(x,0)$$

Example 1.62 Solve the boundary value problem $\dfrac{\partial u}{\partial x}=4\dfrac{\partial u}{\partial y}$, if $u(0,y)=8e^{-3y}$. ∎

Solution: Given that

$$\frac{\partial u}{\partial x}=4\frac{\partial u}{\partial y}, \tag{1.80}$$

with boundary condition $u(0,y)=8e^{-3y}$.
Let the given equation has the solution of the form $u(x,y)=X(x)Y(y)$, where X is function of x alone and Y is function of y alone. Now $\dfrac{\partial u}{\partial x}=X'(x)Y(y)$ and $\dfrac{\partial u}{\partial y}=X(x)Y'(y)$. Putting these values in given equation, we have

$$X'Y=4XY' \implies \frac{X'}{4X}=\frac{Y'}{Y}, \tag{1.81}$$

Since x and y are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X'}{4X} = \frac{Y'}{Y} = k(constant) \implies X' - 4kX = 0 \text{ and } Y' - kY = 0$$

These are ordinary differential equation of first order first degree. Therefore its solutions will be

$$\frac{X'}{X} = 4k \implies \log X = 4kx + \log c_1 \implies \frac{X}{c_1} = 4kx \implies X = c_1 e^{4kx}$$

Similarly solution corresponding to $Y' - kY = 0$, we get $Y = c_2 e^{ky}$. Substituting the values of X and Y in the trail solution $u(x,y) = X(x)Y(y)$ i.e.

$$u(x,y) = c_1 e^{4kx}.c_2 e^{ky} \implies u(x,y) = Ce^{4kx+ky},$$

where $C = c_1 c_2$ is another arbitrary constant.
Now putting $x = 0$ and using boundary condition $u(0,y) = 8e^{-3y}$, we have

$$u(0,y) = Ce^{4k.0+ky} \implies 8e^{-3y} = Ce^{ky}$$

Thus we have $C = 8$ and $k = -3$. Thus the required solution will be $u(x,y) = 8e^{-12x-3y}$.

■ **Example 1.63** Using the method of separation of variable, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 6e^{-3x}$. ■

Solution: Given that

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \tag{1.82}$$

with boundary condition $u(x,0) = 6e^{-3x}$.
Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial u}{\partial x} = X'(x)T(t)$ and $\frac{\partial u}{\partial t} = X(x)T'(t)$. Putting these values in given equation, we have

$$X'T = 2XT' + XT \implies X'T = X(2T' + T) \implies \frac{X'}{X} = 2\frac{T'}{T} + 1, \tag{1.83}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X'}{X} = 2\left(\frac{T'}{T}\right) + 1 = k(constant) \implies X' - kX = 0 \text{ and } 2T' + T - kT = 0$$

These are ordinary differential equation of first order first degree. Therefore its solutions will be

$$X' - kX = 0 \implies \frac{X'}{X} = k \implies \log X = kx + \log c_1 \implies \frac{X}{c_1} = kx \implies X = c_1 e^{kx}$$

Now, solution corresponding to $2T' + T - kT = 0 \implies 2T' = T(k-1) \implies 2\frac{T'}{T} = \frac{(k-1)}{2}$, we get $T = c_2 e^{\frac{(k-1)}{2}t}$. Substituting the values of X and T in the trail solution $u(x,t) = X(x)T(t)$ i.e.

$$u(x,t) = c_1 e^{kx}.c_2 e^{\frac{(k-1)t}{2}} \implies u(x,t) = Ce^{kx + \frac{(k-1)t}{2}},$$

where $C = c_1c_2$ is another arbitrary constant.
Now putting $t = 0$ and using boundary condition $u(x,0) = 6e^{-3x}$, we have

$$u(x,0) = Ce^{kx+\frac{(k-1).0}{2}} \implies 6e^{-3x} = Ce^{kx}$$

Thus we have $C = 6$ and $k = -3$. Thus the required solution will be $u(x,t) = 6e^{-3x-2y}$.

Exercise

Solve the following PDE:

(1.) $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y} = 0$. if $u(0,y) = 8e^{-3y} + 4e^{-5y}$ **Ans.** $u(x,y) = 8e^{-3(4x+y)} + 4e^{-5(4x+y)}$.

(2.) Show that $z(x,y) = 4e^{-3x}\cos 3y$ is a solution to the boundary value problem $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2 y} = 0$, if $z(x,\pi/2) = 0$ and $z(x,0) = 4e^{-3x}$.

(3.) $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t} = 0$ if $u(x,0) = x(4-x)$

(4.) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial u}{\partial y} = 0$ which satisfy the boundary conditions $z = 0$ when $x = 0$ and π; $z = \sin 3x$ when $y = 0$ and $0 < x < \pi$.
Ans. $z(x,y) = \sin 3xe^{-9y}$.

(5.) $2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} = 0$ which satisfy the boundary conditions $0 < x < 3$, $u(0,t) = u(3,t) = 0$ and $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$.
Ans. $u(x,t) = 5e^{-32\pi^2 t}\sin 4\pi x - 3e^{-128\pi^2 t}\sin 8\pi x + 2e^{-200\pi^2 t}\sin 10\pi x$.

General solution of one-dimensional wave (vibrational) equation satisfying the given boundary conditions

Consider one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with boundary conditions $u(0,t) = 0$ and $u(a,t) = 0, \quad \forall t$.
Solution: Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1.84}$$

with boundary conditions $u(0,t) = 0$ and $u(a,t) = 0$.
Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\dfrac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\dfrac{\partial^2 u}{\partial t^2} = X(x)T''(t)$. Putting these values in given equation, we have

$$X''T = \frac{1}{c^2}XT'' \implies \frac{X''}{X} = \frac{T''}{c^2 T}, \tag{1.85}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T''}{c^2T} = k(constant) \implies X'' - kX = 0 \text{ and } T'' - c^2kT = 0$$

These are ordinary differential equation of second order with constant coefficient. Now to solve these two equations $X'' - kX = 0$ and $T'' - c^2kT = 0$, three cases arises:

Case-I When $k = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1x + a_2$$

and

$$T'' = 0 \implies T = a_3t + a_4.$$

Thus the required solution is

$$u(x,t) = (a_1x + a_2)(a_3t + a_4). \tag{1.86}$$

Case-II When $k > 0$, we can take $k = \lambda^2(say)$, then both equations reduces to

$X'' - \lambda^2X = 0 \implies$ the auxiliary equation is $(m^2 - \lambda^2) = 0 \implies m = \pm\lambda$. Therefore its solution will be $X = b_1e^{\lambda x} + b_2e^{-\lambda x}$

and

$$T'' - c^2\lambda^2T = 0 \implies T = b_3e^{c\lambda t} + b_4e^{-c\lambda t}.$$

Thus the required solution is

$$u(x,t) = (b_1e^{\lambda x} + b_2e^{-\lambda x})(b_3e^{c\lambda t} + b_4e^{-c\lambda t}). \tag{1.87}$$

Case-III When $k < 0$, we can take $k = -\lambda^2(say)$, then both equations reduces to

$X'' + \lambda^2X = 0 \implies$ the auxiliary equation is $(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i$. Therefore its solution will be $X = c_1\cos(\lambda x) + c_2\sin(\lambda x)$

and

$$T'' + c^2\lambda^2T = 0 \implies T = c_3\cos(c\lambda t) + c_4\sin(c\lambda t).$$

Thus the required solution is

$$u(x,t) = (c_1\cos(\lambda x) + c_2\sin(\lambda x))(c_3\cos(c\lambda t) + c_4\sin(c\lambda t)). \tag{1.88}$$

Thus the equation (1.86), (1.87) and (1.88) are various possible solution of the given wave equation. Given boundary conditions are $u(0,t) = u(a,t) = 0 \quad \forall t$ In view of the boundary condition, the solution given by the equation (1.86) becomes

$$0 = a_2(a_3t + a_4) \quad \text{and} \quad 0 = (a_1a + a_2)(a_3t + a_2)$$

$$\implies a_2 = 0 \quad \text{and} \quad (a_1a + a_2) = 0 \implies a_1 = a_2 = 0$$

Hence $u(x,t) = 0 \quad \forall t$. This is a trivial solution.

Again, in view of the boundary condition, the solution given by the equation (1.87) becomes

$$0=(b_1+b_2)(b_3e^{c\lambda t}+b_4e^{-c\lambda t}) \quad \text{and} \quad 0=(b_1e^{\lambda a}+b_2e^{-\lambda a})(b_3e^{c\lambda t}+b_4e^{-c\lambda t})$$

$$\implies (b_1+b_2)=0 \quad \text{and} \quad b_1e^{\lambda a}+b_2e^{-\lambda a}=0 \implies b_1=b_2=0$$

Hence $u(x,t)=0 \quad \forall t$. This is also a trivial solution.
Finally, in view of the boundary condition, the solution given by the equation (1.88) becomes

$$0=c_1(c_3\cos(c\lambda t)+c_4\sin(c\lambda t)) \quad \text{and} \quad 0=(c_1\cos(\lambda a)+c_2\sin(\lambda a))(c_3\cos(c\lambda t)+c_4\sin(c\lambda t))$$

$$\implies c_1=0 \quad \text{and} \quad c_2\sin\lambda a=0$$

Now for non-trivial solution of given wave equation, we can not take $c_2=0$

$$\implies \sin\lambda a=0 \implies \lambda a=n\pi \quad n=1,2,3,\ldots$$

Thus $\lambda=\dfrac{n\pi}{a}, n=1,2,3,\ldots$

Hence the solution given by the equation (1.88) becomes

$$u_n(x,t)=c_2\sin\frac{n\pi}{a}\left(c_3\cos\frac{n\pi ct}{a}+c_4\sin\frac{n\pi ct}{a}\right) \quad n=1,2,3,\ldots$$

$$u_n(x,t)=\sin\frac{n\pi}{a}\left(E_n\cos\frac{n\pi ct}{a}+F_n\sin\frac{n\pi ct}{a}\right) \quad n=1,2,3,\ldots$$

Where $E_n=(c_2c_3)$ and $F_n=(c_2c_4)$ are new arbitrary constants.
Since the given wave equation is linear, its most general solution is obtained by applying the principle of superposition, the required solution is

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{a}\left(E_n\cos\frac{n\pi ct}{a}+Fn\sin\frac{n\pi ct}{a}\right) \quad n=1,2,3,\ldots$$

General solution of one-dimensional wave (vibrational) equation satisfying the given boundary and initial conditions

Consider one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where $u(x,t)$ is the deflection of the string. the solution of this equation shows how the string moves. More precisely, if the ends of string are fixed at $x=0$ and $x=a$, we have the two boundary conditions.

$$u(0,t)=0 \text{ and } u(a,t)=0, \quad \forall t.$$

The form of the motion of the string will depend on the initial deflection (deflection at $t=0$) and on the intial velocity (velocity at $t=0$). Denoting the initial deflection by $f(x)$ and initial velocity by $g(x)$, we get two initial conditions

$$u(x,0)=f(x), \qquad 0\le x\le a$$

$$\text{and} \quad \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x), \quad i.e. \quad u_t(x,0)=g(x) \qquad 0\le x\le a.$$

Solution: Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1.89}$$

with boundary conditions $u(0,t) = 0$, $u(a,t) = 0, u(x,0) = f(x)$ and $u_t(x,0) = g(x), \quad 0 \le x \le a$. Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\dfrac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\dfrac{\partial^2 u}{\partial t^2} = X(x)T''(t)$. Putting these values in given equation, we have

$$X''T = \frac{1}{c^2}XT'' \implies \frac{X''}{X} = \frac{T''}{c^2 T}, \tag{1.90}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T''}{c^2 T} = k(constant) \implies X'' - kX = 0 \text{ and } T'' - c^2 kT = 0$$

These are ordinary differential equation of second order with constant coefficient. Now to solve these two equations $X'' - kX = 0$ and $T'' - c^2 kT = 0$, three cases arises:
Case-I When $k = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \ne 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = a_1 x + a_2$ becomes $0 = a_1.0 + a_2$ and $0 = a_1.a + a_2 \implies a_1 = 0 = a_2$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So we reject case-I, when $k = 0$.
Case-II When $k > 0$, we can take $k = \lambda^2 (say)$, then first equations reduces to

$X'' - \lambda^2 X = 0 \implies$ the auxiliary equation is $(m^2 - \lambda^2) = 0 \implies m = \pm\lambda$. Therefore its solution will be $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \ne 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ becomes $0 = b_1 e^{\lambda.0} + b_2 e^{-\lambda.0}$ and $0 = b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies 0 = b_1 + b_2$ and $b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies b_1 = b_2 = 0$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So again we reject case-II, when $k > 0$.
Case-III When $k < 0$, we can take $k = -\lambda^2 (say)$, then first equations reduces to

$X'' + \lambda^2 X = 0 \implies$ the auxiliary equation is $(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i$. Therefore its solution will be $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ becomes $0 = c_1 \cos(\lambda .0) + c_2 \sin(\lambda .0)$ and $0 = c_1 \cos(\lambda a) + c_2 \sin(\lambda a) \implies c_1 = 0$ and $0 = c_2 \sin(\lambda a) = 0$
Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$\implies \sin \lambda a = 0 \implies \lambda a = n\pi \quad n = 1,2,3,...$$

Thus $\lambda = \dfrac{n\pi}{a}, n = 1,2,3,...$
Hence non-zero solution $X_n(x)$ are given by

$$(c_2)_n \sin\left(\frac{n\pi x}{a}\right) \tag{1.91}$$

Similarly the solution corresponding to the equation $T'' + \lambda^2 T = 0$ is

$$T_n(t) = (c_3)_n \cos\frac{n\pi ct}{a} + (c_4)_n \sin\frac{n\pi ct}{a} \tag{1.92}$$

Hence the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(E_n \cos\frac{n\pi ct}{a} + F_n \sin\frac{n\pi ct}{a}\right) \tag{1.93}$$

Where $E_n = ((c_2)_n(c_3))$ and $F_n = ((c_2)_n(c_4)_n)$ are new arbitrary constants.
In order to find a solution which also satisfy $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, We differentiate equation (1.93) w.r.t. t,

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{a}\left(\frac{-n\pi c}{a}E_n \sin\frac{n\pi ct}{a} + \frac{n\pi c}{a}Fn\cos\frac{n\pi ct}{a}\right)\right\} \tag{1.94}$$

Put $t = 0$ in equation (1.93) and (1.94) and using initial equation $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} \tag{1.95}$$

and

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c F_n}{a} \sin\frac{n\pi x}{a} \tag{1.96}$$

Which are Fourier sin series of expansion $f(x)$ and $g(x)$, respectively. Accordingly we get

$$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \tag{1.97}$$

and

$$F_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\frac{n\pi x}{a}dx \tag{1.98}$$

Hence the required solution is given by the equation (1.93) where E_n and F_n are given by the equation (1.97) and (1.98).

■ **Example 1.64** Discuss D'Alembert's solution of one dimensional wave equation. or Show that the general solution of the wave equation

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \text{ is } u(x,t)=\phi(x+ct)+\psi(x-ct),$$

where ϕ and ψ are arbitrary functions. ■

Solution: Given equation is

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Let v and w be two new independent variables such that

$$w=x+ct \qquad and \qquad v=x-ct \tag{1.99}$$

Now

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}$$

Using equation (1.99), we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial w}+\frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial x}=\frac{\partial}{\partial w}+\frac{\partial}{\partial v} \tag{1.100}$$

Thus

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \implies \frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial}{\partial w}+\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial w}+\frac{\partial u}{\partial v}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial w^2}+2\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial v^2} \tag{1.101}$$

Again

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial w}\frac{\partial w}{\partial t}+\frac{\partial u}{\partial v}\frac{\partial v}{\partial t}$$

Using equation (1.99), we have

$$\frac{\partial u}{\partial t}=c\frac{\partial u}{\partial w}-c\frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial t}=c\left(\frac{\partial}{\partial w}-\frac{\partial}{\partial v}\right) \tag{1.102}$$

Thus

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) \implies \frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial}{\partial w}-\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial w}-\frac{\partial u}{\partial v}\right)$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial w^2}-2\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial v^2}\right) \implies \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial w^2}-2\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial v^2}\right) \tag{1.103}$$

Using (1.101) and (1.103) reduces to

$$\frac{\partial^2 u}{\partial w^2}+2\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial v^2}=\frac{\partial^2 u}{\partial w^2}-2\frac{\partial^2 u}{\partial w\partial v}+\frac{\partial^2 u}{\partial v^2}\implies\frac{\partial^2 u}{\partial w\partial v}=0 \tag{1.104}$$

$$\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right)=0 \tag{1.105}$$

Integrating (1.105) w.r.t. v, we get

$$\frac{\partial u}{\partial w}=F(w), \tag{1.106}$$

where F is an arbitrary function of w.
Integrating (1.106) w.r.t. w, we get

$$u=\int F(w)dw+\psi(v),$$

where ψ is an function of v. Then

$$u=\phi(w)+\psi(v),\text{ where }\phi(w)=\int F(w)dw$$

or

$$u=\phi(x+ct)+\psi(x-ct).$$

General solution of one-dimensional heat (diffusion) equation satisfying the given boundary and initial conditions

Consider one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t},$$

where $u(x,t)$ is the temperature of the bar. If both the ends of a bar of length a are at temperature zero and initial temperature is to be prescribed function $f(x)$ in the bar, then find the temperature at a subsequent time t. More precisely, the faces $x=0$ and $x=a$ of an infinite slab are maintained at zero temperature. Given that the temperature $u(x,t)=f(x)$ at $t=0$. Find the temperature at a subsequent time t.
Solution: Given that

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t}, \tag{1.107}$$

with boundary conditions $u(0,t)=0$, $u(a,t)=0$.
The initial condition is given by $u(x,0)=f(x)$, $\quad 0<x<a$
Let the given equation has the solution of the form $u(x,t)=X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial^2 u}{\partial x^2}=X''(x)T(t)$ and $\frac{\partial u}{\partial t}=X(x)T'(t)$. Putting these values in given equation, we have

$$X''T=\frac{1}{k}XT'\implies\frac{X''}{X}=\frac{T'}{kT}, \tag{1.108}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T'}{kT} = \mu(constant) \implies X'' - \mu X = 0 \text{ and } T' - \mu kT = 0$$

These are ordinary differential equation of second order and first order with constant coefficient. Now to solve these two equations

$$X'' - \mu X = 0 \tag{1.109}$$

and

$$T' - \mu kT = 0. \tag{1.110}$$

Now three cases arises:
Case-I When $\mu = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution $u(x,t) = X(x)$T(t) becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = a_1 x + a_2$ becomes $0 = a_1.0 + a_2$ and $0 = a_1.a + a_2 \implies a_1 = 0 = a_2$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So we reject case-I, when $\mu = 0$.
Case-II When $\mu > 0$, we can take $\mu = \lambda^2(say)$, then equations $X'' - \mu X = 0$ reduces to

$$X'' - \lambda^2 X = 0 \implies \text{the auxiliary equation is } (m^2 - \lambda^2) = 0 \implies m = \pm\lambda. \text{ Therefore its solution will be } X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution $u(x,t)X(x)T(t)$ becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ becomes $0 = b_1 e^{\lambda.0} + b_2 e^{-\lambda.0}$ and $0 = b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies 0 = b_1 + b_2$ and $b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies b_1 = b_2 = 0$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So again we reject case-II, when $\mu > 0$.
Case-III When $\mu < 0$, we can take $\mu = -\lambda^2(say)$, then first equations reduces to

$$X'' + \lambda^2 X = 0 \implies \text{the auxiliary equation is } (m^2 + \lambda^2) = 0 \implies m = \pm\lambda i. \text{ Therefore its solution will be } X = c_1\cos(\lambda x) + c_2\sin(\lambda x)$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = c_1\cos(\lambda x) + c_2\sin(\lambda x)$ becomes $0 = c_1\cos(\lambda.0) + c_2\sin(\lambda.0)$ and $0 = c_1\cos(\lambda a) + c_2\sin(\lambda a) \implies c_1 = 0$ and $c_2\sin(\lambda a) = 0$
Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$\implies \sin\lambda a = 0 \implies \lambda a = n\pi \quad n = 1,2,3,\ldots$$

Thus $\lambda = \dfrac{n\pi}{a}, n = 1, 2, 3, \ldots$
Hence non-zero solution $X_n(x)$ are given by

$$X_n(x) = (c_2)_n \sin\left(\frac{n\pi x}{a}\right) \tag{1.111}$$

Now the solution corresponding to the equation $T' + \lambda^2 kT = 0$ is

$$\frac{T'}{T} = -\lambda^2 k \tag{1.112}$$

By integrating we get

$$\log T = -\lambda^2 kt + \log c_3 \implies T = c_3 e^{-\lambda^2 kt} \implies T = c_3 e^{-(n^2\pi^2/a^2)kt} \tag{1.113}$$

Hence solution is $T_n(t) = D_n e^{-C_n^2 t}$, where $C_n = (n^2\pi^2 k/a^2)$ and $D_n = c_3$ are new arbitrary constants.
The general solution is

$$u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-C_n^2 t}, \tag{1.114}$$

where $E_n = (c_2)_n D_n$ is another new arbitrary constants.
Substituting $t = 0$ in (1.114) and using initial condition $u(x,0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) \tag{1.115}$$

Which are Fourier sin series of expansion $f(x)$. Accordingly we get

$$E_n = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a} dx \tag{1.116}$$

Hence the required solution is given by the equation (1.114) and E_n given by the equation (1.116).

Laplace Equation

Definition 1.2.4 A two dimensional Laplace equation is defined as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1.117}$$

and a three dimensional Laplace equation is defined as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \tag{1.118}$$

Laplace equation is also known as **potential equation**.
If the problems involves rectangular boundaries, we use the Laplace equation given by (1.117) and (1.118).

Laplace's Equation in plane polar coordinates

If the given boundary problem involves circular boundaries, we use Laplace's equation in polar coordinates (r,θ).

■ **Example 1.65** Transform the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates (r,θ). ■

Solution: If (x,y) be the Cartesian coordinate's of the point P whose polar coordinates are (r,θ), then

$$x=r\cos\theta \qquad and \qquad y=r\sin\theta \tag{1.119}$$

From (1.119)

$$r^2=x^2+y^2 \qquad and \qquad \theta=\tan^{-1}\frac{y}{x} \tag{1.120}$$

From (1.120)

$$2r\frac{\partial r}{\partial x}=2x \implies \frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta \tag{1.121}$$

and

$$2r\frac{\partial r}{\partial y}=2y \implies \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta \tag{1.122}$$

Also

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r} \tag{1.123}$$

and

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r} \tag{1.124}$$

Now

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \implies \cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \implies \frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2}-\sin\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right)\right]-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}\right.$$
$$\left.+\cos\theta\frac{\partial^2 u}{\partial r\partial \theta}-\frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial \theta}+\sin\theta\frac{\partial^2 u}{\partial \theta^2}\right)\right]$$

Thus

$$\frac{\partial^2 u}{\partial x^2}=\cos^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad (1.125)$$

Again

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \implies \sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \implies \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2}+\cos\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right)\right]+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r}\right.$$
$$\left.+\sin\theta\frac{\partial^2 u}{\partial r\partial \theta}+\frac{1}{r}\left(-\sin\theta\frac{\partial u}{\partial \theta}+\cos\theta\frac{\partial^2 u}{\partial \theta^2}\right)\right]$$

Thus

$$\frac{\partial^2 u}{\partial y^2}=\sin^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad (1.126)$$

Adding (1.125) and (1.126)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Hence Laplace equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

Laplace's equation in cylindrical coordinates

If the given boundary problem involves cylindrical boundaries, we use Laplace's equation in cylindrical coordinates (r, θ, z).

■ **Example 1.66** Transform the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ into polar coordinates (r, θ, z). ■

Solution:

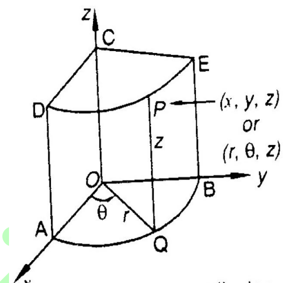


Fig. Cylindrical coordinates

If (x, y, z) be the Cartesian coordinate's of the point P whose cylindrical coordinates are (r, θ, z), then we know that

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad \text{and} \qquad z = z \tag{1.127}$$

With $x = r\cos\theta$, *and* $y = r\sin\theta$, proceed as in the Example (1.65) and prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{1.128}$$

Adding $\dfrac{\partial^2 u}{\partial z^2}$ on both side of (1.128), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{1.129}$$

Hence the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.130}$$

Laplace's Equation in spherical coordinates

If the given boundary problem involves spherical boundaries, we use Laplace's equation in spherical coordinates (r, θ, ϕ).

■ **Example 1.67** Transform the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ into spherical coordinates (r, θ, ϕ). ■

Solution: If (x,y,z) be the Cartesian coordinate's of the point P whose spherical coordinates are (r,θ,ϕ), then

$$x = r\sin\theta\cos\phi \qquad \textit{and} \qquad y = r\sin\theta\sin\phi \qquad \textit{and} \qquad z = r\cos\theta \tag{1.131}$$

From (1.131)

$$r^2 = x^2+y^2+z^2 \qquad \textit{and} \qquad \tan\theta = \frac{(x^2+y^2)^{1/2}}{z} \qquad \textit{and} \qquad \tan\phi = \frac{y}{x}$$

$$r^2 = x^2+y^2+z^2 \qquad \textit{and} \qquad \theta = \tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right) \qquad \textit{and} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.132}$$

From (1.132)

$$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi, \tag{1.133}$$

$$2r\frac{\partial r}{\partial y} = 2y \implies \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi \tag{1.134}$$

And

$$2r\frac{\partial r}{\partial z} = 2z \implies \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta \tag{1.135}$$

Also

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{(x^2+y^2)^{1/2}}{z}\right)^2}\left(\frac{1}{z}\frac{1}{2}\frac{1}{(x^2+y^2)^{1/2}}2x\right) = \frac{\cos\theta\cos\phi}{r}, \tag{1.136}$$

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}, \qquad \textit{and} \qquad \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r} \tag{1.137}$$

And

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \qquad \textit{and} \qquad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \qquad \textit{and} \qquad \frac{\partial\phi}{\partial z} = 0 \tag{1.138}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial x} \implies \sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}$$

$$\implies \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$\begin{aligned}
&= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\\
&+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\\
&-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} &+ \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}\\
&+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi}\\
&+\frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\\
&+\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}
\end{aligned} \tag{1.139}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$\begin{aligned}
&= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\\
&+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\\
&+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)
\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}\\ -\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi}\\ -\frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\\ -\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} \qquad (1.140)\end{aligned}$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

Thus

$$\frac{\partial^2 u}{\partial z^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}. \quad (1.141)$$

Adding (1.139), (1.140) and (1.141)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.$$

Hence Laplace equation in spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

2. Complex Analysis

2.1 Introduction

Complex analysis is the study of functions that live in the complex plane, that is, functions that have complex arguments and complex outputs.This course provides an introduction to complex analysis which is the theory of complex functions of a complex variable. We will start by introducing the complex plane, along with the algebra and geometry of complex numbers, and then we will make our way via differentiation, integration, complex dynamics, power series representation and Laurent series into territories at the edge of what is known today.

2.2 COMPLEX VARIABLE

$x+iy$ is a complex variable and it is denoted by z.

(1) $z = x+iy$ where $i = \sqrt{-1}$ (Cartesian form)
(2) $z = r(\cos\theta + i\sin\theta)$ (Polar form)
(3) $z = re^{i\theta}$ (Exponential form)

2.3 FUNCTIONS OF A COMPLEX VARIABLE

$f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$
$$w = u+iv$$

where u and v are the real and imaginary parts of $f(z)$.

2.4 NEIGHBORHOOD OF Z_0

Let z_0 is a point in the complex plane and let z be any positive number, then the set of points z such that

$$|z - z_0| < \varepsilon$$

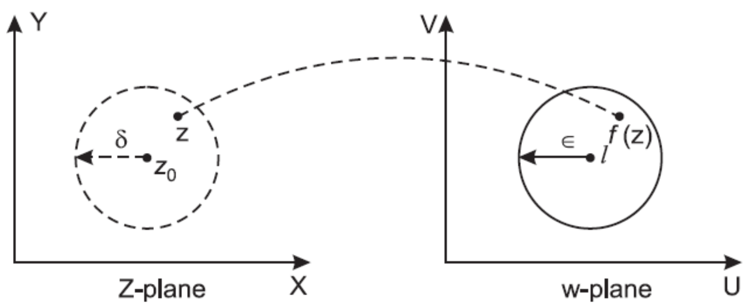
is called $\varepsilon-$neighbourhood of z_0.

2.5 LIMIT OF A FUNCTION OF A COMPLEX VARIABLE

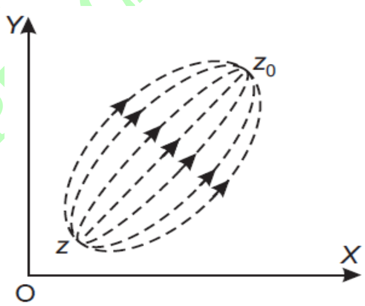
Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\varepsilon > 0$, however small there exists a real number $\delta > 0$, such that

$$|f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

i.e. for every $z \neq z_0$ in δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the ε-disc of w-plane. In symbolic form, $\lim_{z \to z_0} \text{f(z)=l}$.



Note: (I) δ usually depends upon ε.
(II) $z \to z_0$ implies that z approaches z_0 along any path.
The limits must be independent of the manner in which z approaches z_0 If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.



■ **Example 2.1** Prove that $\lim_{z \to 1-i} \dfrac{z^2 + 4z + 3}{z + 1} = 4 - i$

Solution: $\lim_{z \to 1-i} \dfrac{(z+1)(z+3)}{z+1} = \lim_{z \to 1-i} (z+3) = (1-i) + 3 = 4 - i$ ■

■ **Example 2.2** Show that $\lim_{z \to 0} \dfrac{z}{|z|}$ does not exist.

Solution: $\lim_{z \to 0} \dfrac{z}{|z|} = \lim_{(x,y) \to (0,0)} \dfrac{x + iy}{\sqrt{x^2 + y^2}}$

Let $y = mx$, $= \lim_{x \to 0} \dfrac{x + imx}{\sqrt{x^2 + (mx)^2}} = \lim_{x \to 0} \dfrac{1 + im}{\sqrt{1 + (m)^2}} = \dfrac{1 + im}{\sqrt{1 + m^2}}$

The value of $\dfrac{1 + im}{\sqrt{1 + m^2}}$ are different for different value of m. Hence the limit does not exist. ■

■ **Example 2.3** Show that $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist.

Solution: Case-1. $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{(x,y)\to(0,0)} \frac{x+iy}{x-iy} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x+iy}{x-iy}\right] = \lim_{x\to 0} \frac{x}{x} = 1$

Again **Case-2.** $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{(x,y)\to(0,0)} \frac{x+iy}{x-iy} = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x+iy}{x-iy}\right] = \lim_{y\to 0} \frac{iy}{-iy} = -1$

As $z \to 0$ along two different paths, we get different limits. Hence the limit does not exist. ■

Exercise

Show that the limit does not exist

1. $\lim_{z\to 0} \frac{Im(z)^3}{Re(z)^3}$ 2. $\lim_{z\to 0} \frac{z}{(\bar{z})^2}$ 3. $\lim_{z\to 0} \frac{Re(z)^2}{Im(z)}$

Find the limit of the following

5. $\lim_{z\to 0} \frac{Re(z)^2}{|z|}$ **Ans.** 0 6. $\lim_{z\to 1+i} \frac{2z^3}{(Im(z)^2}$ **Ans.** $2(-1+i)$ 7. $\lim_{z\to 0} \frac{z^2+6z+3}{z^2+2z+2}$ **Ans.** $3/2$.

2.6 Continuity

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number ε, we can find a number δ such that $|f(z)-f(z_0)| < \varepsilon$ for all points z of the domain satisfying

$$|z-z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z\to 0} f(z) = f(z_0)$$

■ **Example 2.4** Examine the continuity of the function

$$f(z) = \begin{cases} \dfrac{z^3-iz^2+z-i}{z-i}, & z \neq i \\ 0, & z = i \end{cases}$$

at $z = i$ ■

Solution:

$$\lim_{z\to i} f(z) = \lim_{z\to i} \frac{z^3-iz^2+z-i}{z-i} = \lim_{z\to i} \frac{z^2(z-i)+1(z-i)}{z-i} = \lim_{z\to i} \frac{(z^2+1)(z-i)}{z-i}$$

$$= \lim_{z\to i}(z^2+1) = 0$$

Also, we have $f(i) = 0$. Thus

$$= \lim_{z\to i} f(z) = f(i)$$

Hence $f(z)$ is continuous at $z = i$.

■ **Example 2.5** Show that the function $f(z)$ defined by

$$f(z)=\begin{cases}\dfrac{Re(z)}{z}, z\neq 0\\ 0, \qquad\qquad z=0\end{cases}$$

is not continuous at $z=0$. ■

Solution: Here

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{Re(z)}{z}=\lim_{(x,y)\to(0,0)}\frac{x}{x+iy}=\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x}{x+iy}\right]=\lim_{x\to 0}\frac{x}{x}=1$$

Also

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{Re(z)}{z}=\lim_{(x,y)\to(0,0)}\frac{x}{x+iy}=\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x}{x+iy}\right]=\lim_{y\to 0}\frac{0}{0+iy}=0$$

AS $\lim\limits_{z\to 0}$ for two different paths, limit have two different values. So the limit does not exist. Thus $f(z)$ is not continuous at $z=0$.

Exercise

Examine the continuity of the following functions

(1.) $f(z)=\begin{cases}\dfrac{Im(z)}{|z|}, z\neq 0\\ 0, \qquad\qquad z=0\end{cases}$ at $z=0$. **Ans.** Not Continuous

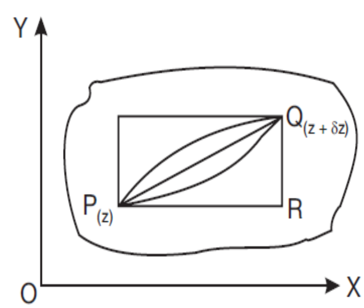
(2.) $f(z)=\dfrac{z^2+3z+4}{z^2+i}$ at $z=1-i$ **Ans.** Continuous

2.7 DIFFERENTIABILITY

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z\to 0$. Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.



■ **Example 2.6** If $f(z)=\begin{cases}\dfrac{x^3y(y-ix)}{x^6+y^2}, z\neq 0\\ 0, \qquad\qquad z=0\end{cases}$ Then discuss $\dfrac{df}{dz}$ at $z=0$. ■

Solution: If $z\to 0$ along radius vector $y=mx$.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \left[\frac{\frac{x^3y(y-ix)}{x^6+y^2} - 0}{x+iy} \right] = \lim_{z\to 0} \left[\frac{-ix^3y(x+iy)}{(x^6+y^2)(x+iy)} \right]$$

$$= \lim_{z\to 0} \left[\frac{-ix^3y}{(x^6+y^2)} \right] = \lim_{x\to 0} \left[\frac{-ix^3(mx)}{(x^6+m^2x^2)} \right] = \lim_{x\to 0} \left[\frac{-imx^2}{(x^4+m^2)} \right] = 0$$

But along $y = x^3$

$$= \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \left[\frac{-ix^3(y)}{(x^6+y^2)} \right] = \lim_{x\to 0} \left[\frac{-ix^3(x^3)}{(x^6+(x^3)^2)} \right] = -\frac{i}{2}$$

In different paths we get different values of $\frac{df}{dz}$ i.e. 0 and $-\frac{i}{2}$. In such a case, the function is not differentiable at $z = 0$.

■ **Example 2.7** Prove that the function $f(z) = |z|^2$ is continuous everywhere but no where differentiable except at the origin. ■

2.8 Analytic Function

Definition 2.8.1 A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.

The point at which the function is not differentiable is called a **singular point** of the function.

An analytic function is also known as "**holomorphic**", "**regular**", "monogenic".

Definition 2.8.2 **Entire Function:** A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

■ **Example 2.8** 1. Polynomials rational functions are entire.
2. $|z|^2$ is differentiable only at $z = 0$. So it is no where analytic. ■

R

1. An entire is always analytic, differentiable and continuous function. But converse is not true.
2. Analytic function is always differentiable and continuous. But converse is not true.
3. A differentiable function is always continuous. But converse is not true

2.9 THE NECESSARY CONDITION FOR $F(Z)$ TO BE ANALYTIC

Theorem 2.9.1 The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{provided} \quad \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \quad \text{and} \quad \frac{\partial v}{\partial x} \text{ exists.}$$

Definition 2.9.1 Cauchy Riemann equations: The equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is known as Cauchy Riemann equations.

2.10 SUFFICIENT CONDITION FOR $F(Z)$ TO BE ANALYTIC

Theorem 2.10.1 The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
2. $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y},$ and $\frac{\partial v}{\partial x}$ are continuous functions of x and y in region R.

■ **Example 2.9** Show that the function $e^x(\cos y + i\sin y)$ is an analytic function, find its derivative.■

Solution: Let $e^x(\cos y + i\sin y) = u + iv$.
So, $e^x\cos y = u$ and $e^x\sin y = v$ then

$$\frac{\partial u}{\partial x} = e^x\cos y, \ \frac{\partial u}{\partial y} = -e^x\sin y, \ \frac{\partial v}{\partial x} = e^x\sin y, \quad \text{and} \quad \frac{\partial v}{\partial y} = e^x\cos y$$

Here we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus are $C-R$ equations and are satisfied and the partial derivatives are continuous. Hence, $e^x(\cos y + i\sin y)$ is analytic.
The derivative of the function $e^x(\cos y + i\sin y)$ is

$$f'(z) = u' + iv' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^x.e^{iy} = e^{x+iy} = e^z$$

■ **Example 2.10** Discuss the analyticity of the function $f(z) = |z|^2$. ■

Solution: $f(z) = |z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$

$$f(z) = x^2 + y^2 = u + iv \implies u = x^2 + y^2, v = 0$$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0}\frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0}\frac{k^2}{k} = 0$$

Also

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,0+k)-v(0,0)}{k} = 0$$

Thus the $C-R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied and the partial derivatives are continuous. Hence, $f(z) = |z|^2$ is analytic at origin.

■ **Example 2.11** Show that the function $f(z) = u + iv$, where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0 \\ 0, \qquad\qquad z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at $z = 0$. Is the function analytic at $z = 0$? Justify your answer. ■

Solution: $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = u + iv$

$$u = \frac{x^3 - y^3}{x^2+y^2}, \quad and \quad v = \frac{x^3+y^3}{x^2+y^2}$$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0} \frac{-k^3/k^2}{k} = -1$$

Also

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,0+k)-v(0,0)}{k} = \lim_{k\to 0} \frac{k^3/k^2}{k} = 1$$

Thus the $C-R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied. Again for derivatives

$$f'(0) = \lim_{z\to 0} \frac{f(z+0)-f(0)}{z} = \lim_{z\to 0} \left[\frac{\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} - 0}{x+iy} \right] = \lim_{z\to 0} \left[\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)} \right]$$

Now let $z \to 0$ along $y = mx$, then

$$= \lim_{x\to 0} \left[\frac{x^3(1+i)-(mx)^3(1-i)}{(x^2+(mx)^2)(x+i(mx))} \right] = \lim_{x\to 0} \left[\frac{(1+i)-(m)^3(1-i)}{(1+(m)^2)(1+im)} \right] = \left[\frac{(1+i)-(m)^3(1-i)}{(1+(m)^2)(1+im)} \right]$$

Which depends on the value of m. So for different paths we get different values of $\frac{df}{dz}$. In such a case, the function is not differentiable at $z = 0$. Hence given function is not analytic at $z = 0$.

2.11 C-R EQUATIONS IN POLAR FORM

The $C-R$ equations in polar form is

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Exercise

Determine which of the following functions are analytic:

(1.) x^2+iy^2 **Ans.** Analytic at all points $y=x$
(2.) $2xy+i(x^{2}\breve{}\,y^{2})$ **Ans.** Not analytic
(3.) $\sin x\cosh y+i\cos x\sinh y$ **Ans.** Yes, analytic
(4.) Show the function of $\bar{z}$ is not analytic any where.
(5.) Discuss the analyticity of the function $f(z)=\begin{cases}\dfrac{x^2y(y-ix)}{x^4+y^2}, z\neq 0\\ 0, \quad z=0\end{cases}$ at $z=0$.

2.12 Harmonic Function

Definition 2.12.1 Any function which satisfies the Laplace's equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

is known as a harmonic function.

Theorem 2.12.1 If $f(z)=u+iv$ is an analytic function, then u and v are both harmonic functions. Such functions u and v are called **Conjugate harmonic functions** if $u+iv$ is also analytic function.

■ **Example 2.12** Prove that $u=x^2-y^2$ and $v=\dfrac{y}{x^2+y^2}$ are harmonic functions of (x,y), but are not harmonic conjugates. ■

Solution:

$$\frac{\partial u}{\partial x}=2x\,,\ \frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial^2 u}{\partial y^2}=-2$$

Thus

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2-2=0.$$

$u(x,y)$ satisfies Laplace equation, hence $u(x,y)$ is harmonic.
Now

$$\frac{\partial v}{\partial x}=\frac{-2xy}{(x^2+y^2)^2},\ \frac{\partial^2 v}{\partial x^2}=\frac{(x^2+y^2)^2(-2y)-(-2xy)2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$=\frac{(x^2+y^2)(-2y)-(-2xy)2(2x)}{(x^2+y^2)^3}=\frac{(6x^2y-2y^3)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2).1 - y.2(x^2+y^2)(2y)}{(x^2+y^2)^2} = \frac{(x^2-y^2)}{(x^2+y^2)^2},$$

$$= \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)2(2y)}{(x^2+y^2)^3} = \frac{(-6x^2y+2y^3)}{(x^2+y^2)^3}$$

Thus

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \left(\frac{(6x^2y-2y^3)}{(x^2+y^2)^3}\right) + \left(\frac{(-6x^2y+2y^3)}{(x^2+y^2)^3}\right) = 0.$$

$v(x,y)$ satisfies Laplace equation, hence $v(x,y)$ is harmonic.
But

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \qquad \textit{and} \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}.$$

Therefore u and v are not harmonic conjugates.

2.13 METHOD TO FIND THE CONJUGATE FUNCTION

Case I. Given. If $f(z) = u + iv$, and u is known.
Claim: We have to find conjugation function v.

■ **Example 2.13** If $w = \phi + i\psi$ represents the complex potential for an electric field and

$$\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

determine the function ϕ. ■

Solution: We have, $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$ so that

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2+y^2).1 - x.(2x)}{(x^2+y^2)^2} = 2x + \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y + \frac{-x.(2y)}{(x^2+y^2)^2} = -2y + \frac{-2xy}{(x^2+y^2)^2}$$

We know that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

Using $C-R$ equations $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, *and* $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$$d\phi = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$

Putting the values of $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$, we get

$$d\phi = \left(-2y + \frac{-2xy}{(x^2+y^2)^2}\right)dx - \left(2x + \frac{(y^2-x^2)}{(x^2+y^2)^2}\right)dy$$

The R.H.S. is an exact differential equation of the form $Mdx + Ndy$. Hence its solution is

$$d\phi = \int \left(-2y + \frac{-2xy}{(x^2+y^2)^2}\right)dx \implies \phi = -2xy + \frac{y}{(x^2+y^2)} + c$$

■ **Example 2.14** Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z. ■

Solution: We have, $u = x^2 - y^2 - 2xy - 2x + 3y$ so that

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$u(x,y)$ satisfies Laplace equation, hence $u(x,y)$ is harmonic.
We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using $C-R$ equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, *and* $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Putting the values of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$, we get

$$dv = -(-2y - 2x + 3)dx + (2x - 2y - 2)dy$$

The R.H.S. is an exact differential equation of the form $Mdx + Ndy$. Hence its solution is

$$v = -\int(-2y - 2x + 3)dx + \int(-2y - 2)dy \implies v = 2xy + x^2 - 3x - y^2 - 2y + c$$

Now,

$$\begin{aligned}
f(z) &= u + iv \\
&= (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2y + c) \\
&= (x^2 - y^2 + 2ixy) + (ix^2 - iy^2 - 2xy) - (2+3i)x - i(2+3i)y + ic \\
&= (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) - (2+3i)x - i(2+3i)y + ic \\
&= (x+iy)^2 + i(x+iy)^2 - (2+3i)(x+iy) + ic \\
&= z^2 + iz^2 - (2+3i)z + ic \\
&= (1+i)z^2 - (2+3i)z + ic
\end{aligned}$$

Which is the required expression of f (z) in terms of z.

■ **Example 2.15** Let $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic function and $u = -r^3 \sin 3\theta$. then construct the corresponding analytic function $f(z)$ in terms of z. ■

Solution: We have $u = -r^3 \sin 3\theta$. Then

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta \text{ and } \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

We know that

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

Using $C-R$ equations in polar form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$$dv = -\frac{1}{r}\frac{\partial u}{\partial \theta} dr + r\frac{\partial u}{\partial r} d\theta$$

Putting the values of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$, we get

$$dv = -\frac{1}{r}(-3r^3 \cos 3\theta)dr + r(-3r^2 \sin 3\theta)d\theta$$

$$dv = (3r^2 \cos 3\theta)dr - (3r^3 \sin 3\theta)d\theta$$

The R.H.S. is an exact differential equation of the form $Mdr + Nd\theta$. Hence its solution is

$$v = \int (3r^2 \cos 3\theta)dr + c \implies v = r^3 \cos 3\theta + c$$

Now,

$$f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic = ir^3(\cos 3\theta + i \sin 3\theta) + ic$$

$$f(z) = ir^3 e^{3i\theta} + ic \implies f(z) = i\left(re^{i\theta}\right)^3 + ic = iz^3 + ic$$

■ **Example 2.16** If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z. ■

Solution: $u + iv = f(z) \implies iu - v = if(z)$
Adding these, $(u-v) + i(u+v) = (1+i)f(z)$ Let

$$U + iV = (1+i)f(z) \text{ where } U = u - v \text{ and } V = u + v$$

$$F(z) = (1+i)f(z)$$

$$U = u - v = (x-y)(x^2 + 4xy + y^2) = x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 \text{ and } \frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

We know that

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$$

Using $C-R$ equations $\dfrac{\partial U}{\partial x}=\dfrac{\partial V}{\partial y}$, *and* $\dfrac{\partial U}{\partial y}=-\dfrac{\partial V}{\partial x}$

$$dV=-\frac{\partial U}{\partial y}dx+\frac{\partial U}{\partial x}dy$$

Putting the values of $\dfrac{\partial U}{\partial x}$ and $\dfrac{\partial U}{\partial y}$, we get

$$dV=-(3x^2-6xy-3y^2)dx+(3x^2+6xy-3y^2)dy$$

The R.H.S. is an exact differential equation of the form $Mdx+Ndy$. Hence its solution is

$$V=-\int(3x^2-6xy-3y^2)dx+\int(-3y^2)dy \implies V=-x^3+3x^2y+3xy^2-y^3+c$$

Now,

$$\begin{aligned}
F(z) &= U+iV\\
&= (x^3+3x^2y-3xy^2-y^3)+i(-x^3+3x^2y+3xy^2-y^3)+ic\\
&= (1-i)x^3+(1+i)3x^2y-(1-i)3xy^2-(1+i)y^3+ic\\
&= (1-i)x^3+i(1-i)3x^2y-(1-i)3xy^2-i(1-i)y^3+ic\\
&= (1-i)[x^3+3ix^2y-3xy^2-iy^3]+ic\\
&= (1-i)(x+iy)^3+ic\\
&= (1-i)z^3+ic
\end{aligned}$$

Thus

$$(1+i)f(z)=(1-i)z^3+ic,$$

$$f(z)=\frac{(1-i)z^3}{(1+i)}+\frac{ic}{(1+i)},$$

Exercise

1.

2.14 MILNE THOMSON METHOD (TO CONSTRUCT AN ANALYTIC FUNCTION)

WORKING RULE: TO CONSTRUCT AN ANALYTIC FUNCTION BY MILNE THOMSON METHOD

Case I. When u is given

Step-1: Find $\dfrac{\partial u}{\partial x}$ and equate it to $\phi_1(x,y)$.

Step-2: Find $\dfrac{\partial u}{\partial y}$ and equate it to $\phi_2(x,y)$.

Step-3: Replace x by z and y by 0 in $\phi_1(x,y)$ to get $\phi_1(z,0)$.

Step-4: Replace x by z and y by 0 in $\phi_2(x,y)$ to get $\phi_2(z,0)$.

Step-5: Find $f(z)$ by the formula $f(z)=\int[\phi_1(z,0)-i\phi_2(z,0)]\,dz+c$

Case II. When v is given

Step-1: Find $\dfrac{\partial v}{\partial x}$ and equate it to $\psi_2(x,y)$.
Step-2: Find $\dfrac{\partial v}{\partial y}$ and equate it to $\psi_1(x,y)$.
Step-3: Replace x by z and y by 0 in $\psi_1(x,y)$ to get $\psi_1(z,0)$.
Step-4: Replace x by z and y by 0 in $\psi_2(x,y)$ to get $\psi_2(z,0)$.
Step-5: Find $f(z)$ by the formula $f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\, dz + c$

■ **Example 2.17** If $u = x^2 - y^2$, find a corresponding analytic function. ■

Solution: Here given that $u = x^2 - y^2$. So that $\dfrac{\partial u}{\partial x} = 2x = \phi_1(x,y)$ and $\dfrac{\partial u}{\partial y} = -2y = \phi_2(x,y)$. On replacing x by z and y by 0, we have

$$\begin{aligned} f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c \\ &= \int (2z) dz + c \\ &= z^2 + c \end{aligned}$$

This is the required analytic function.

■ **Example 2.18** Show that $e^x(x\cos y - y\sin y)$ is a harmonic function. Find the analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part. ■

Solution: Here $v = e^x(x\cos y - y\sin y)$

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \psi_2(x,y)(say), \tag{2.1}$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \psi_1(x,y)(say), \tag{2.2}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y \\ &= e^x(x\cos y - y\sin y + 2\cos y), \end{aligned} \tag{2.3}$$

$$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y). \tag{2.4}$$

Adding equation (2.3) and (2.4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x(x\cos y - y\sin y + 2\cos y) + e^x(-x\cos y + y\sin y - 2\cos y) = 0$$

Hence given function $v = e^x(x\cos y - y\sin y)$ is harmonic function.
Now putting $x = z$ and $y = 0$ in (2.1) and (2.2), we get $\psi_2(z,0) = ze^z + e^z$ and $\psi_1(z,0) = 0$
Hence by Milne-Thomson method, we have

$$\begin{aligned} f(z) &= \int [\psi_1(z,0) + i\psi_2(z,0)]\, dz + c \\ &= \int (0 + i(ze^z + e^z)) dz + c \\ &= i(ze^z - e^z + e^z) + c \\ &= ize^z + c \end{aligned}$$

This is the required analytic function.

2.15 TRANSFORMATION

For every point (x,y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u,v) in the w-plane. We call this **transformation or mapping of z-plane into w-plane**. If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

■ **Example 2.19** Transform the rectangular region $ABCD$ in z-plane bounded by $x=1, x=3$; $y=0$ and $y=3$. Under the transformation $w = z+(2+i)$. ■

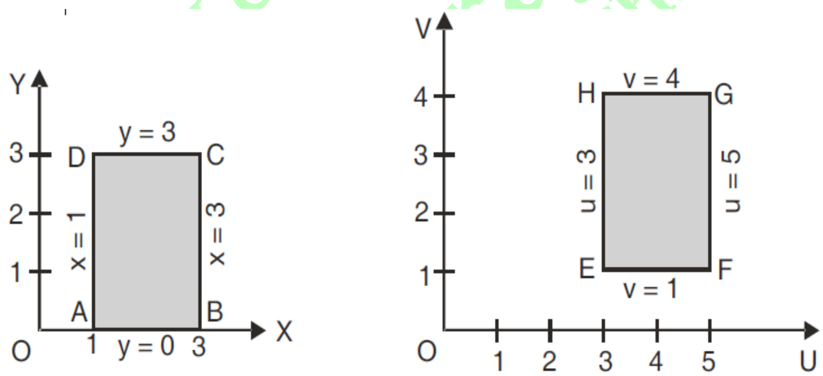
Solution: Here

$$\begin{aligned} w &= z+(2+i) \\ \Longrightarrow u+iv &= x+iy+(2+i) \\ &= (x+2)+i(y+1) \end{aligned}$$

By equating real and imaginary quantities, we have $u = x+2$ and $v = y+1$.

z-plane	w-plane	z-plane	w-plane
x	u = x + 2	y	v = y + 1
1	= 1 + 2 = 3	0	= 0 + 1 = 1
3	= 3 + 2 = 5	3	= 3 + 1 = 4

Here the lines $x=1, x=3$; $y=0$ and $y=1$ in the z-plane are transformed onto the line $u=3, u=5$; $v=1$ and $v=4$ in the w-plane. The region $ABCD$ in z-plane is transformed into the region $EFGH$ in w-plane.



Ans.

■ **Example 2.20** Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$. ■

Solution.

$$\begin{aligned} w &= z^2 \\ \Longrightarrow u+iv &= (x+iy)^2 \\ &= x^2 - y^2 + 2ixy \end{aligned}$$

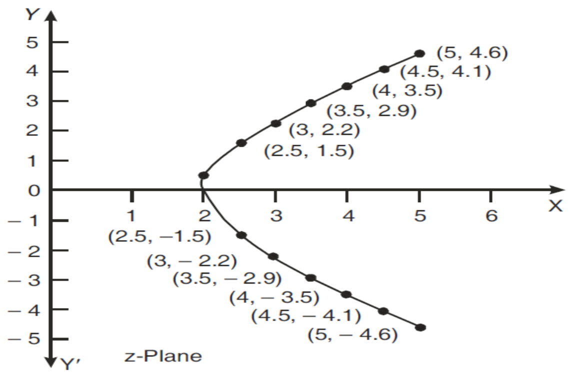
This gives $u = x^2 - y^2$ and $v = 2xy$.

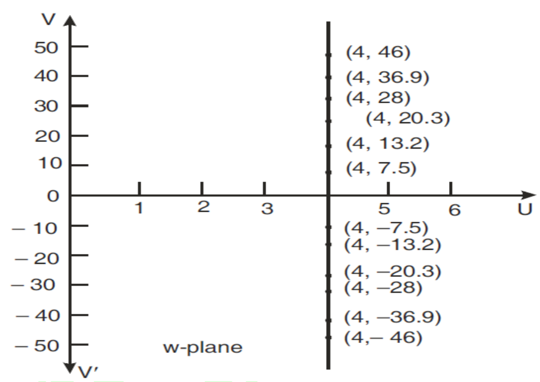
Image of the curve $x^2 - y^2 = 4$ is a straight line, $u = 4$ parallel to the v-axis in w-plane.

Ans.

Table of (x, y) and (u, v)

x	2	2.5	3	3.5	4	4.5	5
$y = \pm\sqrt{x^2 - 4}$	0	± 1.5	± 2.2	± 2.9	± 3.5	± 4.1	± 4.6
$u = x^2 - y^2$	4	4	4	4	4	4	4
$v = 2xy$	0	± 7.5	± 13.2	± 20.3	± 28	± 36.9	± 46





2.16 CONFORMAL TRANSFORMATION

Let two curves C_1, C_2 in the z-plane intersect at the point Z_0 and the corresponding curve C_1^*, C_2^* in the w-plane intersect at $f(z_0)$. **If the angle of intersection of the curves at z_0 in z-plane is the same as the angle of intersection of the curves of w-plane at $f(z_0)$ in magnitude and sense, then the transformation is called conformal**.

If only the magnitude of the angle is preserved, transformation is **Isogonal**.

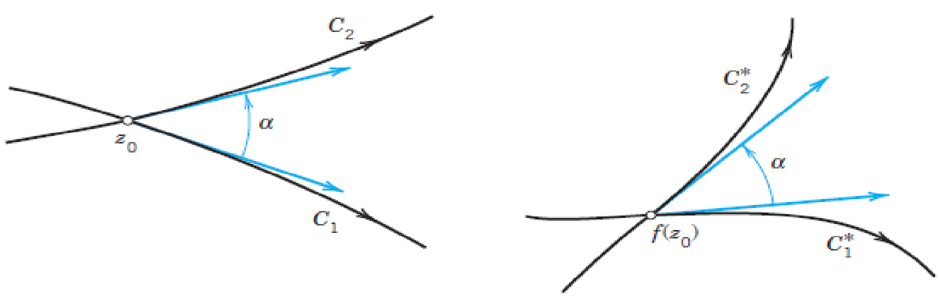


Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

Theorem 2.16.1 If $f(z)$ is analytic, mapping is conformal.

Theorem 2.16.2 Prove that an analytic function $f(z)$ ceases to be conformal at the points where $f'(z) = 0$.

Note 1. The point at which $f'(z) = 0$ is called a **critical point** of the transformation.

■ **Example 2.21** If $u = 2x^2 + y^2$ and $v = \frac{y^2}{x}$, show that the curves $u = constant$ and $v = constant$ cut orthogonally at all intersections but that the transformation $w = u + iv$ is not conformal. ■

Solution: For the curve, $2x^2 + y^2 = u$

$$2x^2 + y^2 = constant = c_1(say) \tag{2.5}$$

Differentiating (2.5), we get

$$4x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2x}{y} = m_1(say) \tag{2.6}$$

For the curve, $\frac{y^2}{x}$ = constant = c_2 (say),

$$y^2 = c_2 x \tag{2.7}$$

Differentiating (2.7), we get

$$2y\frac{dy}{dx} = c_2 \implies \frac{dy}{dx} = \frac{c_2}{2y} = \frac{y^2}{x} \times \frac{1}{2y} = \frac{y}{2x} = m_2(say) \tag{2.8}$$

For orthogonal, from equation (2.6) and (2.8), we have

$$m_1 m_2 = \left(\frac{-2x}{y}\right)\left(\frac{y}{2x}\right) = -1$$

Hence, two curves cut orthogonally.
However, since

$$\frac{du}{dx} = 4x, \ \frac{du}{dy} = 2y, \ \frac{dv}{dx} = -\frac{y^2}{x^2} \text{ and } \frac{dv}{dy} = \frac{2y}{x}$$

The Cauchy-Riemann equations are not satisfied by u and v.
Hence, the function $u + iv$ is not analytic. So, the transformation is not conformal.

■ **Example 2.22** For the conformal transformation $w = z^2$, show that ■

a. The coefficient of magnification at $z = 2 + i$ is $2\sqrt{5}$.
b. The angle of rotation at $z = 2 + i$ is $\tan^{-1}(0.5)$.
c. The coefficient of magnification at $z = 1 + i$ is $2\sqrt{2}$.
d. The angle of rotation at $z = 1 + i$ is $\frac{\pi}{4}$.

Solution:

$$\begin{aligned} w = f(z) &= z^2 \\ \implies f'(z) &= 2z \\ \implies f'(2+i) &= 2(2+i) = 4+2i. \end{aligned}$$

(a.) Coefficient of magnification at $z = 2+i$ is $|f'(2+i)| = |4+2i| = 2\sqrt{5}$.

(b) Angle of rotation at $z = 2+i$ is $ampf'(2+i) = (4+2i) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(0.5)$.

and $f'(1+i) = 2(1+i) = 2+2i$

(c) The coefficient of magnification at $z = 1+i$ is $|f'(1+i)| = |2+2i| = \sqrt{4+4} = 2\sqrt{2}$

(d) The angle of rotation at $z = 1+i$ is $amp.f'(1+i) = 2+2i = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$.

2.17 BILINEAR TRANSFORMATION (Mobius Transformation)

Definition 2.17.1 BILINEAR TRANSFORMATION (Mobius Transformation) The transformation of the form

$$w = \frac{az+b}{cz+d}, \qquad \text{provided } ad - bc \neq 0.$$

is called bilinear transformation.

Definition 2.17.2 INVARIANT POINTS OF BILINEAR TRANSFORMATION We know that

$$w = \frac{az+b}{cz+d},$$

If z maps into itself, then $w = z$

$$z = \frac{az+b}{cz+d}, \tag{2.9}$$

Roots of (2.9) are the invariants or fixed points of the bilinear transformation.
If the roots are equal, the bilinear transformation is said to be parabolic.

Definition 2.17.3 CROSS-RATIO If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)},$$

is called the cross-ratio of z_1, z_2, z_3, z_4.

Theorem 2.17.1 A bilinear transformation preserves cross-ratio of four points i.e.
$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

■ **Example 2.23** Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z| < 1$. ■

Solution: Let the required transformation be $w = \dfrac{az+b}{cz+d}$

$$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z+1} \implies w = \frac{pz+q}{rz+1} \tag{2.10}$$

where $p = \dfrac{a}{d}$, $q = \dfrac{b}{d}$ and $r = \dfrac{c}{d}$.
On substituting the values of $z = 1$ and corresponding values of $w = i$ in (2.10), we get

$$i = \frac{p+q}{r+1} \implies p+q = ir+i \tag{2.11}$$

Again on substituting the values of $z = i$ and corresponding values of $w = 0$ in (2.10), we get

$$0 = \frac{ip+q}{ir+1} \implies ip+q = 0 \tag{2.12}$$

Finally, on substituting the values of $z = -1$ and corresponding values of $w = -i$ in (2.10), we get

$$-i = \frac{-p+q}{-r+1} \implies -p+q = ir-i \tag{2.13}$$

Solving equation (2.11), (2.12) and (2.13, we get $p = i$, $q = 1$ and $r = -i$.
Now substitute the value of p, q and r in (2.10), we get the required Bilinear transformation as

$$w = \frac{iz+1}{-iz+1}. \tag{2.14}$$

To find the image of $|z| < 1$ under the Bilinear map $w = \dfrac{iz+1}{-iz+1}$, we rewrite the given equation in the terms of real and imaginary parts as

$$u+iv = \frac{i(x+iy)+1}{-i(x+iy)+1} = \frac{ix-y+1}{-ix+y+1} = \frac{(ix-y+1)(ix+y+1)}{(-ix+y+1)(ix+y+1)} = \frac{-x^2-y^2+1+2ix}{x^2+(y+1)^2} \tag{2.15}$$

Equating real parts we get

$$u = \frac{-x^2-y^2+1}{x^2+(y+1)^2}. \tag{2.16}$$

But we have, $|z| < 1 \implies x^2+y^2 < 1 \implies 0 < 1-x^2-y^2$. Thus equation (2.16) shows that $u > 0$.
In other words the open disk in z-plane maps into open upper half of w-plane.

2.18 Line Integral

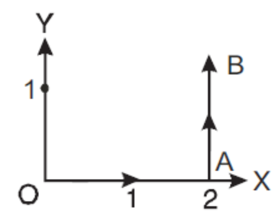
If $f(z) = u(x,y) + iv(x,y)$, then since $dz = dx + idy$,
we have

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy), \text{ where } C \text{ is closed path,}$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

■ **Example 2.24** Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis from $z = 0$ to $z = 2$ and then along a line parallel to y-axis from $z = 2$ to $z = 2+i$. ■

Solution: $\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$
$= \int_{OA} (x)^2 dx + \int_{AB} (2-iy)^2 idy$ Since [Along $OA, y = 0, dy = 0$, x varies 0 to 2. Along $AB, x = 2, dx = 0$ and y varies 0 to 1]
$= \int_0^2 (x)^2 dx + \int_0^1 (2-iy)^2 idy$
$= \int_0^2 x^2 dx + i\int_0^1 (4 - 4iy - y^2)dy$

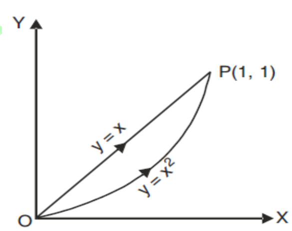


$$= \left[\frac{x^3}{3}\right]_0^2 + i\left[\left(4y - 4i\frac{y^2}{2} - \frac{y^3}{3}\right)\right]_0^1 = \frac{8}{3} + i\left(4 - 4i\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}(14 + 11i).$$

Which is the required value of the given integral.

■ **Example 2.25** Evaluate $\int_0^{1+i} (x^2 - iy)dz$ along the path
(a) $y = x$ (b) $y = x^2$. ■

Solution: (a) Along the line $y = x$,
$dy = dx$ so that $dz = dx + idy$
$dz = dx + idx = (1+i)dx$
By putting $y = x$ and $dz = (1+i)dx$], we have
$\int_0^{1+i} (x^2 - iy)dz = \int_0^1 (x^2 - ix)dx$
$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1+i)\left[\frac{1^3}{3} - i\frac{1^2}{2}\right] = \frac{1}{6}(5-i).$
Which is the required value of the given integral.



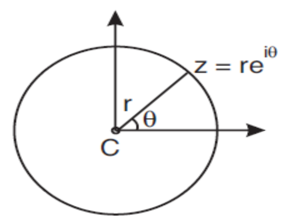
(b) Along the parabola $y = x^2$, $dy = 2xdx$ so that $dz = dx + idy$
$\Longrightarrow$ $dz = dx + 2ixdx = (1+2ix)dx$ and x varies from 0 to 1.
$\int_0^{1+i} (x^2 - iy)dz = \int_0^1 (x^2 - ix^2)(1+2ix)dx$

$$= (1-i)\int_0^1 (x^2 + 2ix^3)dx = (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1 = (1-i)\left[\frac{1^3}{3} + 2i\frac{1^4}{4}\right] = \frac{1}{6}(5+i).$$

Which is the required value of the given integral.

■ **Example 2.26** Evaluate $\int_C (z-a)^n dz$ where C is the circle with centre a and r. Discuss the case when $n = -1$. ■

Solution: The equation of circle C is $|z - a| = r$ or $z - a = re^{i\theta}$ where θ varies from 0 to 2π. so that $dz = rie^{i\theta} d\theta$



By putting $z - a = re^{i\theta}$ and $dz = rie^{i\theta} d\theta$, we have

$\int_C (z-a)^n dz = \int_0^{2\pi} (re^{i\theta})^n rie^{i\theta} d\theta$

$= \int_0^{2\pi} r^{n+1} ie^{i(n\theta+\theta)} d\theta = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right] = \frac{r^{n+1}}{n+1}[\cos(n+1)2\pi + i\sin(n+1)2\pi - 1]$$

$= \dfrac{r^{n+1}}{n+1}[1 + i.0 - 1] = 0.$

When $n = -1$,

$\int_C (z-a)^n dz = \int_C \dfrac{1}{(z-a)} dz = \int_0^{2\pi} \dfrac{1}{re^{i\theta}} rie^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i$

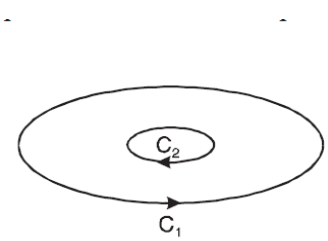
Which is the required value of the given integral.

2.19 IMPORTANT DEFINITIONS

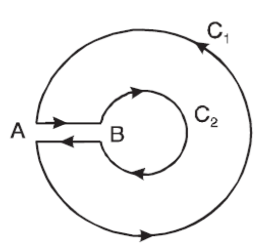
Definition 2.19.1 Simply connected Region: A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region D are the points of the region D.

Definition 2.19.2 Multi-Connected Region: Multi-connected region is bounded by more than one curve. We can convert a multi-connected region into a simply connected one, by giving it one or more cuts.

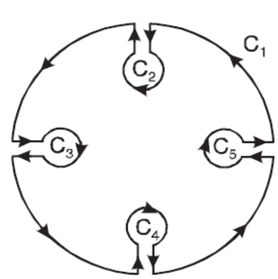
Definition 2.19.3 A function $f(z)$ is said to be meromorphic in a region R if it is analytic in the region R except at a finite number of poles.



Multi-Connected Region



Simply Connected Region



Simply Connected Region

Definition 2.19.4 Single-valued and Multi-valued function: If a function has only one value for a given value of z, then it is a single valued function.

$$\text{For example } f(z) = z^2$$

If a function has more than one value, it is known as multi-valued function,

$$\text{For example } f(z) = \sqrt{z}$$

Definition 2.19.5 Jordan arc: A continuous arc without multiple points is called a Jordan arc.

Definition 2.19.6 Regular arc: If the derivatives of the given function are also continuous in the given range, then the arc is called a regular arc.

Definition 2.19.7 Contour: A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.
The contour is said to be closed if the starting point A of the arc coincides with the end point B of the last arc.

Definition 2.19.8 Zeros of an Analytic function: The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$.

For example,
(1) Zeros of $z^2 - 3z + 2$ are $z = 1$ and $z = 2$.
(2) Zeros of $\cos z$ is $\pm(2n-1)\frac{\pi}{2}$, where $n = 1, 2, 3, \ldots$

Theorem 2.19.1 CAUCHY'S INTEGRAL THEOREM-I If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C, then

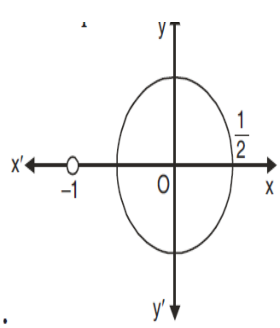
$$\int_C f(z)dz = 0$$

.

Proof: See the proof at page no. 548 in the book written by H.K.Dass
Note: If there is no pole inside and on the contour then the value of the integral of the function is zero.

■ **Example 2.27** Find the integral $\int_C \frac{3z^2+7z+1}{z+1} dz$ where C is the circle $|z| = \frac{1}{2}$ ■

Solution: Poles of the integrand are given by putting the denominator equal to zero.i.e.
$z + 1 = 0 \implies z = -1$ The given circle $|z| = \frac{1}{2}$ with centre at $z = 0$ and radius $\frac{1}{2}$ does not enclose any singularity of the given function. Therefore by Cauchy Integral Formula $\int_C \frac{3z^2+7z+1}{z+1} dz = 0$.



Theorem 2.19.2 **CAUCHY'S INTEGRAL THEOREM-II** If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz = f(a).$$

, where C is any closed curve in R surrounding the point $z=a$.

Proof: See the proof at page no. 551 in the book written by H.K.Dass

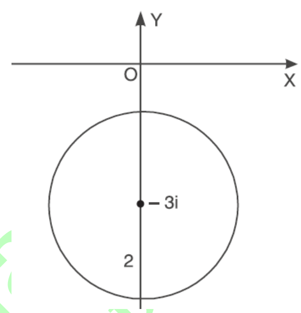
■ **Example 2.28** Evaluate the integral $\int_C \frac{1}{z^2+9}dz$ where C is the circle $|z+3i|=2$ and $|z|=5$. ■

Solution: Here $f(z)=\frac{1}{z^2+9}$.
The poles of f (z) can be determined by equating the denominator equal to zero.

(i.) $z^2+9=0 \implies z=\pm 3i$. Pole at $z=-3i$ lies in the given circle C. $\int_C f(z)dz = \int_C \frac{1}{z^2+9} = \int_C \frac{1}{(z+3i)(z-3i)}$.

$= \int_C \frac{1/(z-3i)}{(z+3i)} = 2\pi i\left[\frac{1}{(z-3i)}\right]_{z=-3i}$

$= 2\pi i\left[\frac{1}{(-3i-3i)}\right] = -\frac{\pi}{3}$.



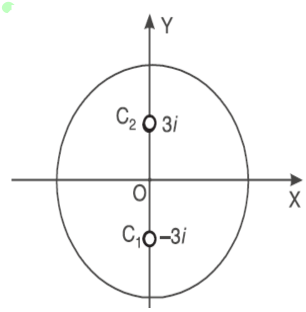
(ii.) $z^2+9=0 \implies z=\pm 3i$. Pole at $z=-3i$ lies in the given circle C. $\int_C f(z)dz = \int_C \frac{1}{z^2+9} = \int_C \frac{1}{(z+3i)(z-3i)}$.

$= \int_C \frac{1/(z-3i)}{(z+3i)} + \int_C \frac{1/(z+3i)}{(z-3i)}$

$= 2\pi i\left[\frac{1}{(z-3i)}\right]_{z=-3i} + 2\pi i\left[\frac{1}{(z+3i)}\right]_{z=3i}$

$= 2\pi i\left[\frac{1}{(-3i-3i)}\right] + 2\pi i\left[\frac{1}{(3i+3i)}\right]$

$= -\frac{\pi}{3}+\frac{\pi}{3}=0.$



Theorem 2.19.3 **CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION** If a function $f(z)$ is analytic in a region R, then its derivative at any point $z=a$ of R is also analytic in R, and is given by,

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz.$$

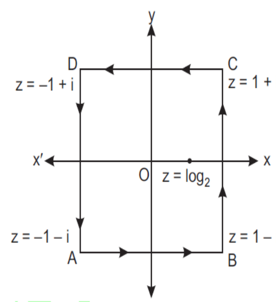
Proof: See the proof at page no. 550 in the book written by H.K.Dass

Theorem 2.19.4 CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF ORDER n OF AN ANALYTIC FUNCTION If a function $f(z)$ is analytic in a region R, then its derivative of order n at any point $z = a$ of R is also analytic in R, and is given by,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz.$$

■ **Example 2.29** Find the integral $\int_C \frac{e^{3z}}{(z-\log 2)^4}dz$, where C is the square with vertices at $\pm 1, \pm i$. ■

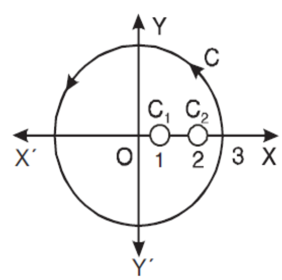
Solution: Here $\int_C \frac{e^{3z}}{(z-\log 2)^4}dz$ Poles of the integrand are given by putting the denominator equal to zero.i.e. $(z-\log 2)^4 = 0 \implies z = \log 2$. The integral has a pole of fourth order.



$$\int_C \frac{e^{3z}}{(z-\log 2)^4}dz = \frac{2\pi i}{3!}f'''\left[e^{3z}\right]_{z=\log 2}$$
$$= \frac{2\pi i}{3!}3.3.3.\left[e^{3z}\right]_{z=\log 2}$$
$$= 9\pi i e^{3\log 2} = 9\pi i e^{\log 2^3} = 9\pi i e^{\log 8} = 72\pi i.$$

■ **Example 2.30** Use Cauchy integral formula to evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz$, where C is the circle $|z| = 3$. ■

Solution: Here $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz$ Poles of the integrand are given by putting the denominator equal to zero.i.e. $(z-1)(z-2) = 0 \implies z = 1, 2$. The integral has two pole at $z = 1, 2$. The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the poles $z = 1$, and $z = 2$.

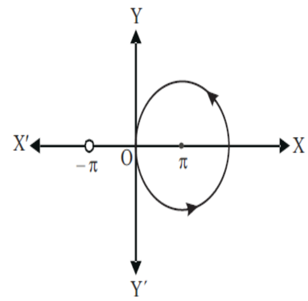


$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz$$
$$= \int_{C_1} \frac{(\sin\pi z^2 + \cos\pi z^2)/(z-2)}{(z-1)}dz + \int_{C_2} \frac{(\sin\pi z^2 + \cos\pi z^2)/(z-1)}{(z-2)}dz$$
$$= 2\pi i\left[\frac{(\sin\pi z^2 + \cos\pi z^2)}{(z-2)}\right]_{z=1} + 2\pi i\left[\frac{(\sin\pi z^2 + \cos\pi z^2)}{(z-1)}\right]_{z=2}$$
$$= 2\pi i\left[\frac{(\sin\pi + \cos\pi)}{(1-2)}\right] + 2\pi i\left[\frac{(\sin 4\pi + \cos 4\pi)}{(2-1)}\right] = 2\pi i\left[\frac{-1}{(-1}\right] + 2\pi i\left[\frac{1}{1}\right] = 4\pi i.$$

Which is the required value of the given integral.

■ **Example 2.31** Use Cauchy integral formula to evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz$, where C is the circle $|z-\pi| = 3.2$. ■

Solution: Here $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz$, where C is a circle $|z-\pi| = 3.2$ with centre π and radius 3.2. Poles of the integrand are given by putting the denominator equal to zero.i.e. $(z+\pi)^3 = 0 \implies z = -\pi, -\pi, -\pi$. The integral has a pole of order 3 at $z = \pi$. But there is no pole within C. By Cauchy Integral Formula $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz = 0$.



Which is the required value of the given integral.

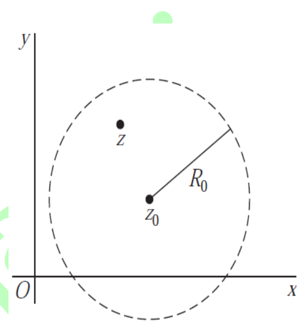
2.20 Taylor's Theorem

Theorem: Suppose that a function $f(z)$ is analytic throughout a disk $|z-a| < R$, centered at a and with radius R. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad (|z-a| < R)$$

where

$$a_n = \frac{f^n(a)}{n!} \qquad (n = 0, 1, 2, \ldots).$$



i.e. $f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots + \frac{(z-a)^n}{n!}f^n(a) + \ldots$. This series is called **Taylor's Series** of $f(z)$ about $z = a$.

If $a = 0$, then the series $f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^n}{n!}f^n(0) + \ldots$ is called **Maclaurin's Series** of $f(z)$ about $z = 0$.

■ **Example 2.32** Obtain the Taylor's series expansion of the function $f(z) = \frac{1}{z^2+(1+2i)z+2i}$ about $z = 0$. ■

Solution: Here the given function is $f(z) = \frac{1}{z^2+(1+2i)z+2i}$. This function can be written as

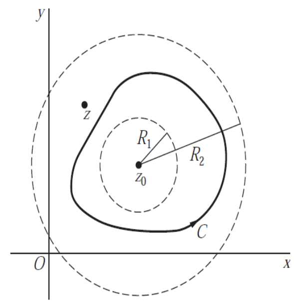
$$f(z) = \frac{1}{(z+2i)(z+1)} = \frac{1}{(2i-1)(z+1)} + \frac{1}{(1-2i)(z+2i)} = \frac{1}{(2i-1)}(z+1)^{-1} + \frac{1}{(1-2i)}(z+2i)^{-1}$$

$$= \frac{1}{(2i-1)}(z+1)^{-1} + \frac{1}{2i(1-2i)}\left(1+\frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{(2i-1)}\left[1-z+z^2-z^3+\ldots\right] + \frac{1}{(2i+4)}\left[1-\frac{z}{2i}-\frac{z^2}{4}-\frac{z^3}{8i}+\ldots\right]$$

After simplifying we get the required expansion.

Theorem: Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z-a| < R_2$, centered at a, and let C denote any positively oriented simple closed contour around a and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation



$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$
$$(R_1 < |z-a| < R_2)$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad (n = 0, 1, 2, ...).$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}dz \qquad (n = 1, 2, 3, ...).$$

2.21 Laurent's Theorem

Definition 2.21.1 Laurent's series: An expansion of the function $f(z)$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

is called Laurent's series expansion. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ is called **Principal Part** of the function $f(z)$ at $z = 0$.

■ **Example 2.33** Obtain the Laurent's series expansion of the function $f(z) = \dfrac{1}{(z+1)(z+3)}$, which is valid for

(a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$. ■

Solution: Here the given function is $f(z) = \dfrac{1}{(z+1)(z+3)}$. Resolving this function into partial fractions, we get

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right).$$

(a) For $1 < |z| < 3$**:**
Since $|z| > 1$ and $|z| < 3$, the above fractions can be written as

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2z}\left(\frac{1}{1+1/z}\right) - \frac{1}{2}\frac{1}{3}\left(\frac{1}{1+z/3}\right).$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}.$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + ...\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + ...\right].$$

(b) For $|z| > 3$:

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2z}\left(\frac{1}{1+1/z}\right) - \frac{1}{2z}\left(\frac{1}{1+3/z}\right).$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}.$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + ...\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + ...\right].$$

$$= \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + ...\right] - \frac{1}{2}\left[\frac{1}{z} - \frac{3}{z^2} + \frac{3^2}{z^3} - \frac{3^3}{z^4} + ...\right].$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - ....$$

(c) For $|z+1| < 2$:

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+1+2}\right).$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(1+\frac{z+1}{2}\right)^{-1} = \frac{1}{2(z+1)} - \frac{1}{4}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + ...\right]$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - ....$$

■ **Example 2.34** Expands $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$, in $1 < |z| < 2$. ■

Solution: Here the given function is $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$, where $1 < |z| < 2$ or $1 < |z|^2 < 4$. Resolving this function into partial fractions, we get

$$f(z) = \frac{z}{5}\left(\frac{1}{z^2-1} - \frac{1}{z^2+4}\right)$$

Since $|z|^2 > 1$ and $|z|^2 < 4$, the above fractions can be written as

$$\frac{z}{5z^2}\left(\frac{1}{1-1/z^2}\right) - \frac{z}{5.4}\left(\frac{1}{1+z^2/4}\right) = \frac{1}{5z}\left(1-\frac{1}{z^2}\right)^{-1} - \frac{z}{20}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{5z}\left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + ...\right] - \frac{z}{20}\left[1 - \frac{z^2}{4} + \frac{z^4}{4^2} - \frac{z^6}{4^3} + ...\right].$$

$$= \frac{1}{5}\left[\frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + ...\right] - \frac{1}{20}\left[z - \frac{z^3}{4} + \frac{z^5}{4^2} - \frac{z^7}{4^3} + ...\right].$$

Which is the Laurent's expansion of $f(z)$in $1<|z|<2$.

■ **Example 2.35** Expands $f(z)=\dfrac{7z-2}{(z+1)z(z-2)}$, in $1<|z+1|<3$. ■

Solution: Let $z+1=u$, then the given function is $f(z)=\dfrac{7z-2}{(z+1)z(z-2)}$ can be written as $f(u)=\dfrac{7(u-1)-2}{u(u-1)(u-1-2)}=\dfrac{7u-9}{u(u-1)(u-3)}$, where $1<|u|<3$. Resolving this function into partial fractions, we get

$$f(u)=-\frac{3}{u}+\frac{1}{u-1}+\frac{2}{u-3}$$

Since $|u|>1$ and $|u|<3$, the above fractions can be written as

$$-\frac{3}{u}+\frac{1}{u}\left(\frac{1}{1-1/u}\right)+\frac{1}{3}\left(\frac{2}{u/3-1}\right)=-\frac{3}{u}+\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$=-\frac{3}{u}+\frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+...\right)-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\frac{u^3}{3^3}+...\right)$$

$$=-\frac{2}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\frac{1}{u^4}+...-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+\frac{u^3}{3^3}+...\right)$$

$$=-\frac{2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\frac{1}{(z+1)^4}+...-\frac{2}{3}\left(1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\frac{(z+1)^3}{27}+...\right).$$

Which is the Laurent's expansion of $f(z)$in $1<|z+1|<3$.

2.22 SINGULARITIES OF ANALYTIC FUNCTION

Definition 2.22.1 A zero of analytic function $f(z)$ is the value of z for which $f(z)=0$.

Definition 2.22.2 SINGULAR POINT: A point at which a function $f(z)$ is not analytic is known as a singular point or singularity of the function.

For Example: The function $\dfrac{1}{z-a}$ has a singular point at $z-a=0$ or $z=a$.

Definition 2.22.3 Isolated singular point: If $z=a$ is a singularity of $f(z)$ and if there is no other singularity in the neighborhood of the point $z=a$, then $z=a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called **non-isolated**.

For Example: The function $\dfrac{1}{(z-a)(z-b)}$ has a singular point at $z=a,b$. Here in the neighborhood of a and b, there does not exits any other singularities. Hence a and b are isolated singularities.

Example of non-isolated singularity: The function $f(z)=cosec\left(\dfrac{\pi}{z}\right)$ is not analytic at the points where $\sin\left(\dfrac{\pi}{z}\right)=0$ i.e., at the points $\dfrac{\pi}{z}=n\pi$ i.e., the points $z=\dfrac{1}{n}$ i.e., the points $z=1,\dfrac{1}{2},\dfrac{1}{3},\dfrac{1}{4},....$

Here $z=0$ is the limit points of $z=\dfrac{1}{n}$. Hence $z=0$ is the non-isolated singularity of the function

$f(z) = cosec\left(\frac{\pi}{z}\right)$ because in the neighbourhood of z = 0, there are infinite number of other singularities $z = \frac{1}{n}$, when n is very large.

Definition 2.22.4 Pole: If the principle part of the function $f(z)$ at $z = a$ in Laurent's expansion has only finite number of terms (say m), we say $f(z)$ has pole of order m at $z = a$. or
if $\exists$ a +ve integer m such that

$$\lim_{z \to a}(z-a)^m f(z) = k(constant) \neq 0.,$$

then we say that $f(z)$ has a pole of order m at $z = a$.

For Example:1. The function $f(z) = \frac{1}{(z-1)^2(z+2)^5}$ has a pole at $z = 1$ of order 2 and has a pole at $z = -2$ of order 5.
2. $\tan z$ and $\sec z$ has simple poles at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots.$
3. $\cot z$ and $cosecz$ has simple poles at $z = 0, \pm\pi, \pm 2\pi, \ldots.$

Definition 2.22.5 Essential Singularities: If the principle part of $f(z)$ at $z = a$ in Laurent's series expansion has infinite number of terms, then we say that $z = a$ is an essential singularities of $f(z)$. or
If $\lim_{z \to a} f(z)$ does not exist, then we say that $z = a$ is essential singularities $f(z)$.

For Example:1. The function $f(z) = e^{1/z}$ has an essential singularities at $z = 0$ because its expansion about $z = 0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

has infinite number of terms in negative powers of z.
2. The function $f(z) = \sin\left(\frac{1}{z-a}\right)$ has an essential singularities at $z = a$ because its expansion about $z = a$

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \ldots$$

has infinite number of terms in negative powers of $z - a$.

Definition 2.22.6 Removable Singularities: If the principle part of $f(z)$ at $z = a$ in Laurent's series expansion has no terms, then we say that $z = a$ is a removable singularities of $f(z)$. or
$z = a$ is said to be removable singularities if $\lim_{z \to a} f(z)$ exist finitely.

For Example:1. The function $f(z) = \frac{\sin z}{z}$ has removal singularities at $z = 0$ because its expansion about $z = 0$

$$\frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 \ldots$$

has no number of terms in negative powers of z.
2. The function $f(z) = \frac{z - \sin z}{z^2}$ has a removable singularities at $z = 0$ because its expansion about $z = 0$

$$\frac{z-\sin z}{z^2} = \frac{1}{z^2}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}+\ldots\right)\right] = \frac{z}{3!}-\frac{z^3}{5!}+\ldots$$

has no number of terms in negative powers of z.

■ **Example 2.36** Find out the zeros and discuss the nature of the singularities of

$$f(z) = \frac{z-2}{z^2}\left(\sin\frac{1}{z-1}\right).$$

■

Solution: Poles of $f(z)$ are given by equating to zero the denominator of $f(z)$ i.e. $z=0$ is a pole of order two.

zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ i.e., $(z-2)\sin\left(\frac{1}{z-1}\right)=0$

$\Longrightarrow$ Either $z-2=0$ or $\sin\left(\frac{1}{z-1}\right)=0$

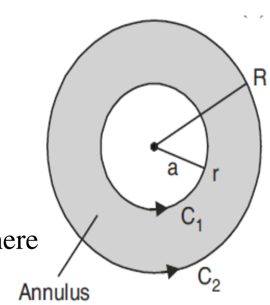
$\Longrightarrow$ $z=2$ and $\frac{1}{z-1}=n\pi$

$\Longrightarrow$ $z=2$ and $z=1+\frac{1}{n\pi}, n=\pm1,\pm2,\pm3,\ldots$

Thus, $z=2$ is a simple zero. The limit point of the zeros $z=1+\frac{1}{n\pi}$ are given by $z=1$. Hence $z=1$ is an isolated essential singularity.

2.23 DEFINITION OF THE RESIDUE AT A POLE

Let $z=a$ be a pole of order m of a function $f(z)$ and C_1 circle of radius r with centre at $z=a$ which does not contain any other singularities except at $z=a$ then $f(z)$ is analytic within the annulus $r<|z-a|<R$ can be expanded within the annulus. Laurent's series:



$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$, where

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{(n+1)}}dz \qquad \text{[h!]}$$

and

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{(-n+1)}}dz$$

$|z-a|=r$ being the circle C_1.
Particularly,

$$b_1 = \frac{1}{2\pi i}\int_{C_1}f(z)dz.$$

The coefficient b_1 is called residue of $f(z)$ at the pole $z=a$. It is denoted by symbol $Res.(z=a)=b_1$.

2.24 RESIDUE AT INFINITY

Residue of $f(z)$ at $z=\infty$ is defined as

$$-\frac{1}{2\pi i}\int_C f(z)dz,$$

where the integration is taken round C in anti-clockwise direction. where C is a large circle containing all finite singularities of $f(z)$.

2.25 METHOD OF FINDING RESIDUES

a. **Residue at simple pole:** (i.) If $f(z)$ has a simple pole at $z=a$, then

$$Res.f(a)=\lim_{z\to a}(z-a)f(z)$$

(ii.) If $f(z)$ is of the form $f(z)=\frac{\phi(z)}{\psi(z)}$, where $\psi(a)=0$ but $\phi(a)\neq 0$. then

$$Res.(z=a)=\frac{\phi(a)}{\psi'(a)}$$

b. **Residue at a pole of order n.** If $f(z)$ has a pole of order n at $z=a$, then

$$Res.(z=a)=\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

c. **Residue at a pole $z=a$ of any order (simple or of order n)**

$Resf(a)=$ coefficient of $\frac{1}{t}$.

Rule. Put $z=a+t$ in the function $f(z)$, expand it in powers of t. Coefficient of $\frac{1}{t}$ is the residue of $f(z)$ at $z=a$.

d. **Residue at a pole $z=\infty$**

$Resf(z=\infty)=\lim_{z\to\infty}[-zf(z)]$.

or The residue of $f(z)$ at infinity $=-\frac{1}{2\pi i}\int_C f(z)dz$.

■ **Example 2.37** Find the residue at $z=0$ of $z\cos\frac{1}{z}$. ■

Solution: Expanding the function in powers of $\frac{1}{z}$, we have

$$z\cos\frac{1}{z}=z\left[1-\frac{1}{2!z^2}+\frac{1}{4!z^4}-...\right]=z-\frac{1}{2!z}+\frac{1}{4!z^3}-...$$

This is the Laurent's expansion about $z=0$.
The coefficient of $\frac{1}{z}$ is $-\frac{1}{2}$. So the residue of $z\cos\frac{1}{z}$ at $z=0$ is $-\frac{1}{2}$.

■ **Example 2.38** Find the residue of $f(z)=\frac{z^3}{z^2-1}$ at $z=\infty$. ■

Solution: We have, $f(z) = \dfrac{z^3}{z^2-1}$

$$= \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\ldots\right) = z+\frac{1}{z}+\frac{1}{z^3}+\ldots$$

Residue at infinity =-Coefficient of $\frac{1}{z} = -1$.

■ **Example 2.39** Evaluate the residues of $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$ at $z = 1,2,3$ and ∞ and show that their sum is zero. ■

Solution: Here $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$. The poles of $f(z)$ are determined by putting the denominator equal to zero.

$$(z-1)(z-2)(z-3) = 0 \implies z = 1,2,3$$

Residue of $f(z)$ at $(z=1)$

$$= \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z^2}{(z-1)(z-2)(z-3)} = \lim_{z\to 1}\frac{z^2}{(z-2)(z-3)} = \frac{1}{2}$$

Again, Residue of $f(z)$ at $(z=2)$

$$= \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z^2}{(z-1)(z-2)(z-3)} = \lim_{z\to 2}\frac{z^2}{(z-1)(z-3)} = -4$$

Also, Residue of $f(z)$ at $(z=3)$

$$= \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}(z-3)\frac{z^2}{(z-1)(z-2)(z-3)} = \lim_{z\to 3}\frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$$

Finally, Residue of $f(z)$ at $(z=\infty)$

$$= \lim_{z\to\infty} -zf(z) = \lim_{z\to\infty}\frac{(-z)z^2}{(z-1)(z-2)(z-3)} = \lim_{z\to\infty}\frac{-1}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)} = -1$$

Sum of the residues at all the poles of $f(z) = \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$. Hence, the sum of the residues is zero.

■ **Example 2.40** Find the residue of $f(z) = \dfrac{1}{(z^2+1)^3}$ at $z = i$. ■

Solution: Here $f(z) = \dfrac{1}{(z^2+1)^3}$. The poles of $f(z)$ are determined by putting denominator equal to zero. i.e.

$$(z^2+1)^3 = 0 \implies (z-i)^3(z+i)^3 = 0 \implies z = \pm i$$

Here, $z = i$ is a pole of order 3 of $f(z)$.
Residue at $z = i$

$$= \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\left[(z-i)^3\frac{1}{(z-i)^3(z+i)^3}\right]\right\}_{z=i} = \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]\right\}_{z=i}$$

$$\frac{1}{2!}\left[\frac{3\times 4}{(z+i)^5}\right]_{z=i} = \frac{1}{2}\frac{12}{(i+i)^5} = \frac{6}{2^5 i^5} = \frac{-3i}{16}$$

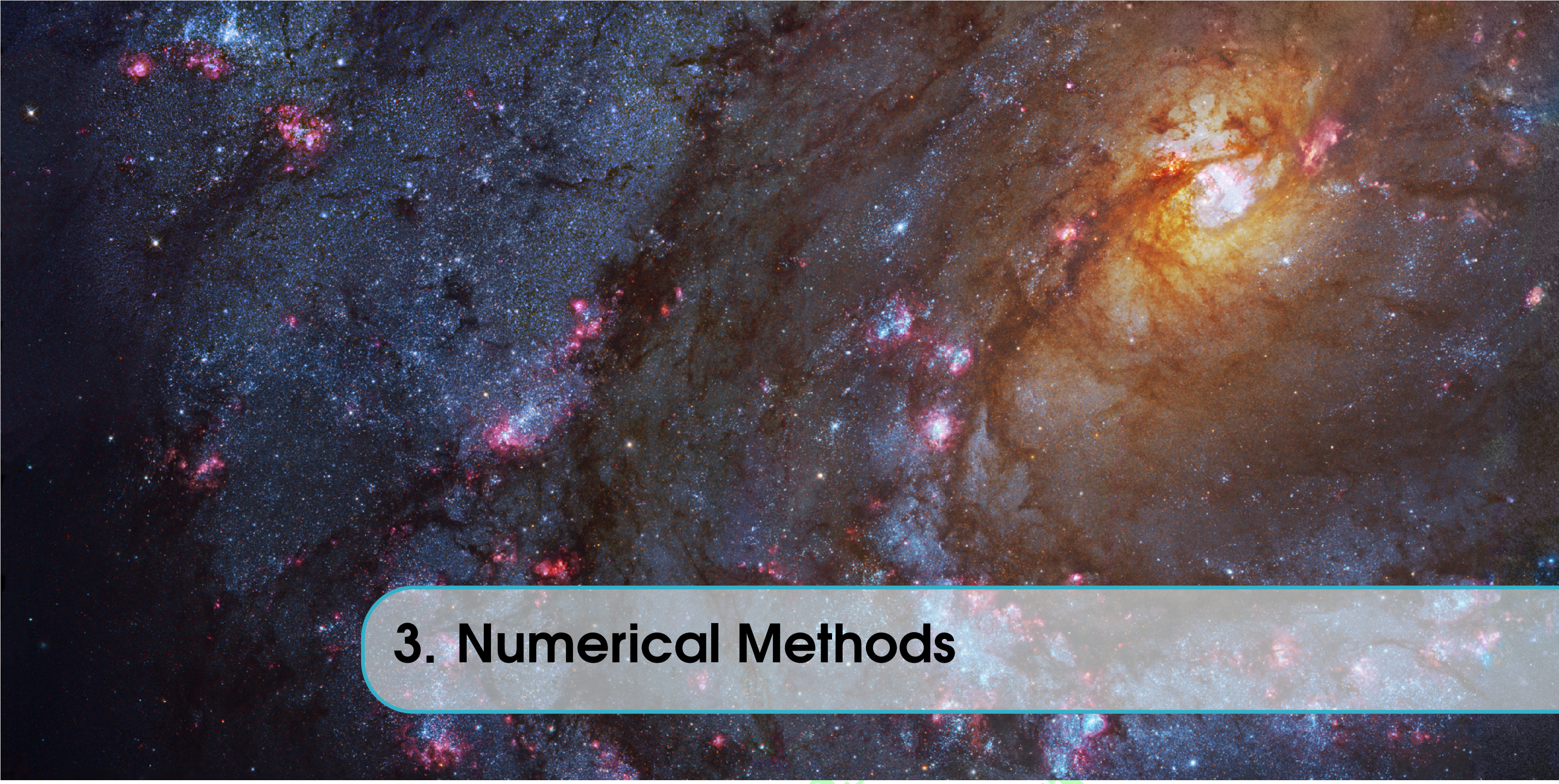
Hence, the residue of the given function at $z = i$ is $\frac{-3i}{16}$.

■ **Example 2.41** Determine the poles and residue at each pole of the function $f(z) = \cot z$. ■

Solution: Here $f(z) = \cot z = \frac{\cos z}{\sin z}$. The poles of the function $f(z)$ are given by

$$\sin z = 0 \implies z = n\pi, \text{ where } n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Residue of $f(z)$ at $z = n\pi$ is $\frac{\cos z}{\frac{d}{dz}(\sin z)} = \frac{\cos z}{\cos z} = 1.$ $\quad \left[Res.(z=a) = \frac{\phi(a)}{\psi'(a)}\right]$



3. Numerical Methods

Definition 3.0.1 A polynomial equation of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 = 0$$

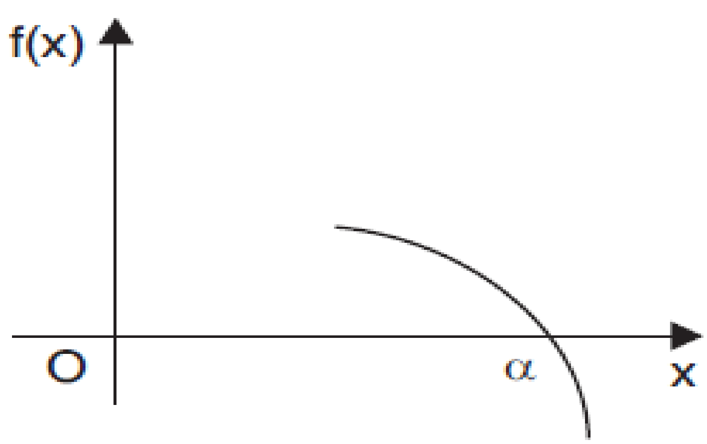
is called an **algebraic equation**.

For Example: $3x^5 + 2x^3 - x^2 + 35 = 0$, $x^4 + 5x^2 + 7 = 0$, $-2x^2 - 3^x + 4 = 0$,

Definition 3.0.2 An equation which contains polynomials, exponential functions, logarithmic functions, trigonometric functions etc. is called a **transcendental equation**.

For Example: $xe^x - 2x = 0$, $x\tan x - \log x = 4$, $\sin^2 x + \cos x = 0$ are transcendental equations.

Definition 3.0.3 **Root/zero:** A number α, for which $f(\alpha) \equiv 0$ is called a root of the equation $f(x) = 0$, or a zero of $f(x)$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x-axis.



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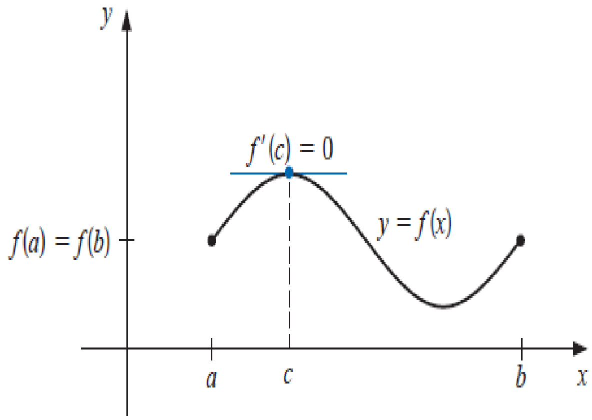
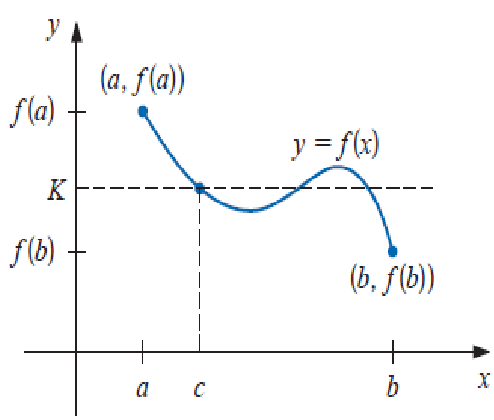
Theorem 3.0.1 Suppose the function f is continuous in $[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number c in (a,b) exists with $f'(c) = 0$.



Theorem 3.0.2 **Intermediate Value Theorem:** If $f(x)$ is continuous on some interval $[a,b]$ and $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots in the interval (a,b).



■ **Example 3.1** Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $(0,1)$. ■

Solution: Consider the function defined by $x^5 - 2x^3 + 3x^2 - 1 = 0$. The function f is continuous on $[0,1]$. In addition, Here $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Therefore by, Intermediate Value Theorem there exist a number x with $0 < x < 1$, for which $x^5 - 2x^3 + 3x^2 - 1 = 0$. Hence the given function has the solution in the interval $(0,1)$.

Bisection Method This method is applicable for numerically solving the equation $f(x) = 0$ for the real variable x, where f is a continuous function defined on an interval $[a,b]$ and $f(a)$ and $f(b)$

have opposite signs. Then by the intermediate value theorem, the continuous function f must have at least one root in the interval (a,b). Now at each step, this method divides the interval in two interval by computing the midpoint $c = (a+b)/2$ of the interval and the value of the function f at that point c. Unless c is itself a root, there are now only two possibilities: either $f(a)$ and $f(c)$ have opposite signs or $f(c)$ and $f(b)$ have opposite signs. The method selects the subinterval that is guaranteed to be a root in the new interval. The process is continued until the interval is sufficiently small. Explicitly, if $f(a)$ and $f(c)$ have opposite signs, then the method sets c as the new value for b, and if $f(b)$ and $f(c)$ have opposite signs then the method sets c as the new value for a. (If $f(c) = 0$ then c may be taken as the solution and the process stops.) In both cases, the new $f(a)$ and $f(b)$ have opposite signs, so the method is applicable to this smaller interval.

■ **Example 3.2** Find the root of the equation $x^3 - x - 1 = 0$ by bisection method up to two places of decimal. ■

Solution: Here $f(x) = x^3 - x - 1$. Let $x_0 = 0$ so that $f(x_0 = 0) = -1 < 0$. and $x_1 = 2$ so that $f(x_1 = 2) = (2)^3 - (2) - 1 = 5 > 0$. Thus by intermediate value theorem the roots lies in the interval $(0,2)$. By using bisection method, the first approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0+2}{2} = 1$$

Now $f(x_2 = 1) = (1)^3 - (1) - 1 = -1 < 0$. Since $f(x_1 = 2)f(x_2 = 1) = 5.(-1) = -5 < 0$. Therefore the roots lies in the interval $(x_2, x_1) i.e. (1,2)$. Again by using bisection method, the second approximation is

$$x_3 = \frac{x_2 + x_1}{2} = \frac{1+2}{2} = \frac{3}{2}$$

Now $f\left(x_3 = \frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right) - 1 = \frac{7}{8} > 0$. Since $f(x_2 = 1)f(x_3 = 3/2) = (-1).(7/8) = -7/8 < 0$. Therefore the roots lies in the interval $(x_2, x_3) i.e. (1, 3/2)$. Again by using bisection method, the third approximation is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{1+3/2}{2} = \frac{5}{4}$$

Now $f\left(x_4 = \frac{5}{4}\right) = \left(\frac{5}{4}\right)^3 - \left(\frac{5}{4}\right) - 1 = -\frac{19}{64} < 0$. Since $f(x_4 = 5/4)f(x_3 = 3/2) = (-19/64).(7/8) = 133/512 > 0$. Therefore the roots lies in the interval $(x_4, x_3) i.e. (5/4, 3/2)$. Repeating this process we get $x_5 = 11/8, x_6 = 21/16, x_7 = 43/32, x_8 = 85/64$. This process will be continue until the difference between last two approximation is less than 0.005.

■ **Example 3.3** Using bisection method, find the root of the equation $3x - \sqrt{1+\sin x} = 0$. ■

Solution: Here $f(x) = 3x - \sqrt{1+\sin x}$. Let $x_0 = 0$ so that $f(x_0 = 0) = -1 < 0$. and $x_1 = 1$ so that $f(x_1 = 1) = 3(1) - \sqrt{1+\sin(1)} = 3 - \sqrt{1+0.84} = 3 - 1.35 = 1.65 > 0$. Thus by intermediate value theorem the roots lies in the interval $(0,1)$. By using bisection method, the first approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0+1}{2} = 1/2 = 0.5$$

Now $f(x_2 = 0.5) = 3(0.5) - \sqrt{1+\sin(0.5)} = 3 - \sqrt{1+0.479} = 1.5 - 1.216 = 0.28 > 0$. Since $f(x_0 = 0) f(x_2 = 0.5) = (-1)(0.28) = -0.28 < 0$. Therefore the roots lies in the interval $(x_0, x_2) i.e. (0, 0.5)$. Again by using bisection method, the second approximation is

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0+0.5}{2} = 0.25$$

Now $f(x_3 = 0.25) = 3(0.25) - \sqrt{1+\sin(0.25)} = 0.75 - \sqrt{1+0.247} = 0.75 - 1.216 = -0.117 < 0$. Since $f(x_2 = 0.5) f(x_3 = 0.25) = (0.28)(-0.117) = -0.33 < 0$. Therefore the roots lies in the interval $(x_2, x_3) i.e. (0.5, 0.25)$. Again by using bisection method, the third approximation is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0.5+0.25}{2} = 0.35$$

Continuing this process we get the required answer.

Quiz

Question 1: What is a continuous function?
Question 2: How the midpoint is calculated in the Bisection method?
Question 3: What is a root of a function?
Question 4: After applying one iteration, by how much did our interval that might contain a zero of f decrease?

1. Almost half
2. More than half
3. 50%
4. 70%

Regula-Falsi method: At the start of all iterations of the method, we require the interval in which the root lies. Let the root of the equation $f(x) = 0$, lie in the interval (x_{k-1}, x_k), that is, $f_{k-1} f_k < 0$, where $f(x_{k-1}) = f_{k-1}$, and $f(x_k) = f_k$. Then, $P(x_{k-1}, f_{k-1})$, $Q(x_k, f_k)$ are points on the curve $f(x) = 0$.

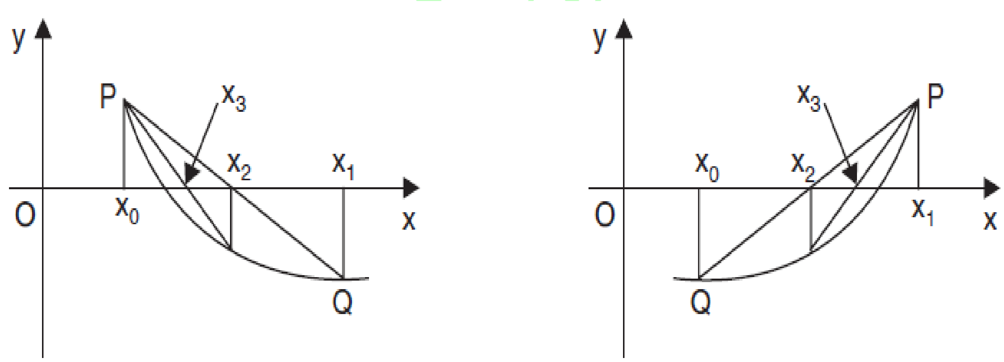


Figure 3.1: Regula-falsi method

Draw a straight line joining the points P and Q. The line PQ is taken as an approximation of the curve in the interval $[x_{k-1}, x_k]$. The equation of the line PQ is given by

$$\frac{y-f_k}{f_{k-1}-f_k} = \frac{x-x_k}{x_{k-1}-x_k}$$

The point of intersection of this line PQ with the $x-$axis is taken as the next approximation to the root. Setting $y=0$, and solving for x, we get

$$x = x_k - \left(\frac{x_{k-1}-x_k}{f_{k-1}-f_k}\right) f_k = x_k - \left(\frac{x_k-x_{k-1}}{f_k-f_{k-1}}\right) f_k$$

Thus the $(k+1)^{th}$ iteration will be

$$x_{k+1} = x_k - \left(\frac{x_k-x_{k-1}}{f_k-f_{k-1}}\right) f_k = \frac{x_{k-1}f_k - x_k f_{k-1}}{f_k - f_{k-1}}$$

This method is also called **linear interpolation method** or **chord method** or **false position method**.

■ **Example 3.4** Locate the intervals which contain the positive real roots of the equation $x^3-3x+1=0$. Obtain these roots correct to three decimal places, using the method of false position. ■

Solution: We form the following table of values for the function $f(x)$.

x	0	1	2	3
$f(x)$	1	-1	3	19

There is one positive real root in the interval $(0,1)$ and another in the interval $(1,2)$. There is no real root for $x>2$ as $f(x)>0$, for all $x>2$.
First, we find the root in $(0,1)$. We have $x_0=0, x_1=1, f_0=f(x_0)=f(0)=1, f_1=f(x_1)=f(1)=-1$.

$$x_2 = \frac{x_0f_1-x_1f_0}{f_1-f_0} = \frac{(0)(-1)-(1)(1)}{(-1)-(1)} = \frac{1}{2} = 0.5$$

Now, $f_2=f(x_2)=f(0.5)=-0.375$. Since, $f(0)f(0.5)<0$, the root lies in the interval $(0,0.5)$.

$$x_3 = \frac{x_0f_2-x_2f_0}{f_2-f_0} = \frac{(0)(-0.375)-(0.5)(1)}{(-0.375)-(1)} = 0.36364$$

Now, $f_3=f(x_3)=f(0.36364)=-0.04283$. Since, $f(0)f(0.36364)<0$, the root lies in the interval $(0,0.36364)$.

$$x_4 = \frac{x_0f_3-x_3f_0}{f_3-f_0} = \frac{(0)(-0.04283)-(0.36364)(1)}{(-0.04283)-(1)} = 0.34870$$

Now, $f_4=f(x_4)=f(0.34870)=-0.00370$. Since, $f(0)f(0.34870)<0$, the root lies in the interval $(0,0.34870)$.

$$x_5 = \frac{x_0f_4-x_4f_0}{f_4-f_0} = \frac{(0)(-0.00370)-(0.34870)(1)}{(-0.00370)-(1)} = 0.34741$$

Now, $f_5=f(x_5)=f(0.34741)=-0.00030$. Since, $f(0)f(0.34741)<0$, the root lies in the interval $(0,0.34741)$.

$$x_6 = \frac{x_0f_5-x_5f_0}{f_5-f_0} = \frac{(0)(-0.00030)-(0.34741)(1)}{(-0.00030)-(1)} = 0.347306$$

Now, $|x_6-x_5| = |0.34730-0.34741| = 0.0001 < 0.0005$.
The root has been computed correct to three decimal places. The required root can be taken as $x=x_6=0.347306$. We may also give the result as 0.347, even though x_6 is more accurate. Note that the left end point $x=0$ is fixed for all iterations.
Now, we compute the root in $(1,2)$. We have
$x_0=1, x_1=2, f_0=f(x_0)=f(1)=-1, f_1=f(x_1)=f(2)=3$.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(1)(3) - (2)(-1)}{(3) - (-1)} = 1.25$$

Now, $f_2 = f(x_2) = f(1.25) = -0.796875$. Since, $f(1.25)f(2) < 0$, the root lies in the interval $(1.25, 2)$.

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(1.25)(3) - (2)(-0.796875)}{(3) - (-0.796875)} = 1.407407$$

Now, $f_3 = f(x_3) = f(1.407407) = -0.434437$. Since, $f(1.407407)f(2) < 0$, the root lies in the interval $(1.407407, 2)$.
Similarly, we get $x_4 = 1.482367$, $x_5 = 1.513156$, $x_6 = 1.525012$, $x_7 = 1.529462$, $x_8 = 1.531116$, $x_9 = 1.531729$, $x_{10} = 1.531956$.
Now, $|x_{10} - x_9| = |1.531956 - 1.53179| = 0.000227 < 0.0005$.
The root has been computed correct to three decimal places. The required root can be taken as $x = x_{10} = 1.531956$. Note that the right end point $x = 2$ is fixed for all iterations.

■ **Example 3.5** Find the root correct to two decimal places of the equation $xe^x = \cos x$, using the method of false position. ■

Solution: Define $f(x) = \cos x - xe^x = 0$. We form the following table of values for the function $f(x)$.

x	0	1
$f(x)$	1	-2.17798

A root of the equation lies in the interval $(0, 1)$. Let $x_0 = 0, x_1 = 1$. Using the method of false position, we obtain the following results.

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(0)(-2.17798) - (1)(1)}{(-2.17798) - (1)} = 0.31467$$

Now, $f_2 = f(x_2) = f(0.31467) = 0.51986$. Since, $f(0.31467)f(1) < 0$, the root lies in the interval $(0.31467, 1)$.

$$x_3 = \frac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = \frac{(0.31467)(-2.17798) - (1)(0.51986)}{(-2.17798) - (0.51986)} = 0.44673$$

Now, $f_3 = f(x_3) = f(0.44673) = 0.20354$. Since, $f(0.44673)f(1) < 0$, the root lies in the interval $(0.44673, 1)$.

$$x_4 = \frac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = \frac{(0.44673)(-2.17798) - (1)(0.20354)}{(-2.17798) - (0.20354)} = 0.49402$$

Now, $f_4 = f(x_4) = f(0.49402) = 0.07079$. Since, $f(0.49402)f(1) < 0$, the root lies in the interval $(0.49402, 1)$.

$$x_5 = \frac{x_4 f_1 - x_1 f_4}{f_1 - f_4} = \frac{(0.49402)(-2.17798) - (1)(0.07079)}{(-2.17798) - (0.07079)} = 0.50995$$

Now, $f_5 = f(x_5) = f(0.50995) = 0.02360.$. Since, $f(0.50995)f(1) < 0$, the root lies in the interval $(0.50995, 1)$.
Similarly we get, $x_6 = 0.51520$, $x_7 = 0.51692$.
Now, $|x_7 - x_6| = |0.51692 - 0.51520| = 0.00172 < 0.005$. The root has been computed correct to two decimal places. The required root can be taken as $x = x_7 = 0.51692$. Note that the right end point $x = 2$ is fixed for all iterations.

Quiz

Question 1: Write the method of Regula- falsi method to obtain a root of $f(x) = 0$?
Question 2: What is the disadvantage of the Regula- falsi method ?
Question 3: Find the smallest positive root of $x = e^{2x}$, correct to two decimal places using Regula-falsi method ?

3.1 Newton-Raphson method:

Let a root of $f(x) = 0$ lie in the interval (a, b). Let x_0 be an initial approximation to the root in this interval. The Newton-Raphson method to find this root is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \text{ provided } f'(x_k) \neq 0$$

This method is called the Newton-Raphson method or simply the **Newton's method**. The method is also called the **tangent method**.

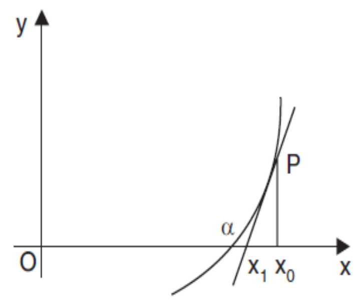


Figure 3.2: Newton-Raphson method

■ **Example 3.6** Perform four iterations of the Newton's method to find the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$. ■

Solution: We have $f(0) = 1$, $f(1) = -3$. Since, $f(0)f(1) < 0$, the smallest positive root lies in the interval $(0, 1)$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5}, \qquad k = 0, 1, 2, \ldots$$

Let $x_0 = 0.5$. We have the following results.

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(0.5)^3 - 1}{3(0.5)^2 - 5} = 0.176471,$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(0.176471)^3 - 1}{3(0.176471)^2 - 5} = 0.201568,$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 5} = \frac{2(0.201568)^3 - 1}{3(0.201568)^2 - 5} = 0.201640,$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 5} = \frac{2(0.201640,)^3 - 1}{3(0.201640,)^2 - 5} = 0.201640.$$

Therefore, the root correct to six decimal places is $x \equiv 0.201640$.

■ **Example 3.7** Using Newton-Raphson method solve $x\log_{10}x = 12.34$ with $x_0 = 10$. ■

Solution: Here $f(x) = x\log_{10}x - 12.34$. Then $f'(x) = \log_{10}x + \frac{1}{\log_e 10} = \log_{10}x + 0.434294$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k\log_{10}x_k - 12.34}{\log_{10}x_k + 0.434294} = \frac{(0.434294)x_k + 12.34}{\log_{10}x_k + 0.434294}, \qquad k = 0,1,2,\ldots$$

Let $x_0 = 10$. We have the following results.

$$x_1 = \frac{(0.434294)x_0 + 12.34}{\log_{10}x_0 + 0.434294} = \frac{(0.434294)(10) + 12.34}{\log_{10}10 + 0.434294} = 11.631465.$$

$$x_2 = \frac{(0.434294)x_1 + 12.34}{\log_{10}x_1 + 0.434294} = \frac{(0.434294)(11.631465) + 12.34}{\log_{10}(11.631465) + 0.434294} = 11.594870.$$

$$x_3 = \frac{(0.434294)x_2 + 12.34}{\log_{10}x_2 + 0.434294} = \frac{(0.434294)(11.594870) + 12.34}{\log_{10}(11.594870) + 0.434294} = 11.594854.$$

We have $|x_3 - x_2| = |11.594854 - 11.594870| = 0.000016$. Therefore, We may take $x \equiv 11.594854$ as the root correct to four decimal places.

■ **Example 3.8** Derive the Newton's method for finding $1/N$, where $N > 0$. Hence, find $1/17$, using the initial approximation as (i) 0.05, (ii) 0.15. Do the iterations converge ■

Solution: Let $x = \frac{1}{N} \implies N = \frac{1}{x}$. Define a function $f(x) = \frac{1}{x} - N$ so that $f'(x) = -\frac{1}{x^2}$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{\frac{1}{x_k} - N}{\left(-\frac{1}{x_k^2}\right)} = x_k + \left[x_k - Nx_k^2\right] = 2x_k - Nx_k^2, \qquad k = 0,1,2,\ldots$$

(i) With $N = 17$, and $x_0 = 0.05$, we obtain the sequence of approximations

$$x_1 = 2x_0 - Nx_0^2 = 2(0.05) - 17(0.05)^2 = 0.0575.$$

$$x_2 = 2x_1 - Nx_1^2 = 2(0.0575) - 17(0.0575)^2 = 0.058794.$$

$$x_3 = 2x_2 - Nx_2^2 = 2(0.058794) - 17(0.058794)^2 = 0.058823.$$

$$x_4 = 2x_3 - Nx_3^2 = 2(0.058823) - 17(0.058823)^2 = 0.058823.$$

Since, $|x_4 - x_3| = 0$, the iterations converge to the root. The required root is 0.058823.
(ii) With $N = 17$, and $x_0 = 0.15$, we obtain the sequence of approximations

$$x_1 = 2x_0 - Nx_0^2 = 2(0.15) - 17(0.15)^2 = -0.0825.$$

$$x_2 = 2x_1 - Nx_1^2 = 2(-0.0825.) - 17(-0.0825.)^2 = -0.280706.$$

$$x_3 = 2x_2 - Nx_2^2 = 2(-0.280706) - 17(-0.280706)^2 = -1.900942.$$

$$x_4 = 2x_3 - Nx_3^2 = 2(-1.900942) - 17(-1.900942)^2 = -65.23275.$$

We find that $x_k \to -\infty$ as k increases. Therefore, the iterations diverge very fast. This shows the importance of choosing a proper initial approximation.

■ **Example 3.9** Derive the Newton's method for finding the q^{th} root of a positive number N, $N^{1/q}$, where $N > 0, q > 0$. Hence, compute $17^{1/3}$ correct to four decimal places, assuming the initial approximation as $x_0 = 2$. ■

Solution: Let $x = N^{1/q} \implies N = x^q$. Define a function $f(x) = x^q - N$ so that $f'(x) = qx^{q-1}$. Applying the Newton's method, we obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^q - N}{qx^{q-1}} = \frac{(q-1)x_k^q + N}{qx^{q-1}}, \qquad k = 0, 1, 2, \ldots$$

For computing $17^{1/3}$, we have $q = 3$ and $N = 17$. Hence, the method becomes

$$x_{k+1} = \frac{(3-1)x_k^3 + 17}{3x_k^{3-1}} = \frac{2x_k^3 + 17}{3x_k^2}, \qquad k = 0, 1, 2, \ldots$$

With $x_0 = 2$, we obtain the following results.

$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = \frac{2(2)^3 + 17}{3(2)^2} = 2.75,$$

$$x_2 = \frac{2x_1^3 + 17}{3x_1^2} = \frac{2(2.75)^3 + 17}{3(2.75)^2} = 2.582645,$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = \frac{2(2.582645)^3 + 17}{3(2.582645)^2} = 2.571332,$$

$$x_4 = \frac{2x_3^3 + 17}{3x_3^2} = \frac{2(2.571332)^3 + 17}{3(2.571332)^2} = 2.571282.$$

Since, $|x_4 - x_3| = |2.571282 - 2.571332| = 0.00005.$, We may take $x = 2.571282$ as the required root correct to four decimal places.

Quiz

Question 1: The Newton-Raphson method formula for finding the square root of a real number N from the equation $x^2 - N = 0$ is,

1. $x_{i+1} = \frac{x_i}{2}$
2. $x_{i+1} = \frac{3x_i}{2}$
3. $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$
4. $x_{i+1} = \frac{1}{2}\left(3x_i - \frac{N}{x_i}\right)$

Question 2: Evaluate $\sqrt{142}$, correct to three decimal places ?

Question 3: Write an iteration formula for finding the value of $1/N$, where N is a real number.?

3.2 Difference Operator

3.2.1 Interpolation with equally spaced data

Let the data $(x_i, f(x_i))$ be given with uniform spacing, that is, the nodal points are given by $x_i = x_0 + ih, i = 0, 1, 2, \ldots, n$. Now we define several finite difference operators and relation between these finite difference operators.

Notation: We use the following notations as follows:

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_i = x_0 + ih, \text{ and}$$
$$f_0 = f(x_0), f_1 = f(x_1), f_2 = f(x_2), \ldots, f_i = f(x_i), \ldots$$

Definition 3.2.1 Shift Operator E: The Shift operator E is defined as

$$Ef(x) = f(x+h)$$

In particular, $Ef(x_0) = f(x_0+h) = f(x_1), Ef(x_1) = f(x_0+2h) = f(x_2), \ldots, Ef(x_i) = f(x_0+(i+1)h) = f(x_{i+1}), \ldots$

Therefore, the operator E when applied on $f(x)$ shifts it to the value at the next nodal point. We have

$$E^2 f(x) = E\left(Ef(x)\right) = E\left(f(x+h)\right) = f(x+2h).$$

In general, we have

$$E^k f(x) = f(x+kh), \text{ where } k \text{ is any real number.}$$

For example: $E^{1/2}[f(x)] = f(x + \frac{1}{2}h)$.

Definition 3.2.2 Forward Operator Δ: The forward operator Δ is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

In particular,

$$\begin{aligned}
\Delta f(x_0) &= f(x_0+h) - f(x_0) = f(x_1) - f(x_0), \\
\Delta f(x_1) &= f(x_0+2h) - f(x_0+h) = f(x_2) - f(x_1), \\
&\cdot \\
&\cdot \\
&\cdot \\
\Delta f(x_i) &= f(x_0+(i+1)h) - f(x_0+ih) = f(x_{i+1}) - f(x_i),
\end{aligned}$$

These differences are called the first forward differences.

The second forward difference is defined by

$$\begin{aligned}
\Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x)) = \Delta f(x+h) - \Delta f(x) \\
&= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\} \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}$$

The forward differences can be written in a tabular form as in Table 3.1

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	$f(x_0)$	$\Delta f(x_0) =$ $f(x_1) - f(x_0)$			
x_1	$f(x_1)$	$\Delta f(x_1) =$ $f(x_2) - f(x_1)$	$\Delta^2 f(x_0) =$ $\Delta f(x_1) - \Delta f(x_0)$	$\Delta^3 f(x_0) =$ $\Delta^2 f(x_1) - \Delta^2 f(x_0)$	
x_2	$f(x_2)$	$\Delta f(x_2) =$ $f(x_3) - f(x_2)$	$\Delta^2 f(x_1) =$ $\Delta f(x_2) - \Delta f(x_1)$	$\Delta^3 f(x_1) =$ $\Delta^2 f(x_2) - \Delta^2 f(x_1)$	$\Delta^4 f(x_0) =$ $\Delta^3 f(x_1) - \Delta^3 f(x_0)$
x_3	$f(x_3)$	$\Delta f(x_3) =$ $f(x_4) - f(x_3)$	$\Delta^2 f(x_2) =$ $\Delta f(x_3) - \Delta f(x_2)$		
x_4	$f(x_4)$				

Table 3.1: Forward Difference Table

■ **Example 3.10** Construct the forward difference table for the data

x	-1	0	1	2
$f(x)$	-8	3	1	12

■

Solution: We have the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-8			
		$3-(-8)=11$		
0	3		$-2-11=-13$	
		$1-3=-2$		$13+13=26$
1	1		$11+2=13$	
		$12-1=11$		
2	12			

Table 3.2: Forward Difference Table

Definition 3.2.3 Backward Difference Operator ∇: The Backward difference operator ∇ is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

In particular,

$$\begin{aligned}
\nabla f(x_1) &= f(x_0+h) - f(x_0) = f(x_1) - f(x_0), \\
\nabla f(x_2) &= f(x_0+2h) - f(x_0+h) = f(x_2) - f(x_1), \\
&\;. \\
&\;. \\
&\;. \\
\nabla f(x_{i+1}) &= f(x_0+(i+1)h) - f(x_0+ih) = f(x_{i+1}) - f(x_i),
\end{aligned}$$

These differences are called the first backward differences.

The second backward difference is defined by

$$\begin{aligned}
\nabla^2 f(x) &= \nabla(\nabla f(x)) = \nabla(f(x) - f(x-h)) = \nabla f(x) - \nabla f(x-h) \\
&= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\} \\
&= f(x) - 2f(x-h) + f(x-2h)
\end{aligned}$$

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
x_0	$f(x_0)$	$\nabla f(x_0) = f(x_1) - f(x_0)$			
x_1	$f(x_1)$	$\nabla f(x_1) = f(x_2) - f(x_1)$	$\nabla^2 f(x_0) = \nabla f(x_1) - \nabla f(x_0)$	$\nabla^3 f(x_0) = \nabla^2 f(x_1) - \nabla^2 f(x_0)$	
x_2	$f(x_2)$	$\nabla f(x_2) = f(x_3) - f(x_2)$	$\nabla^2 f(x_1) = \nabla f(x_2) - \nabla f(x_1)$	$\nabla^3 f(x_1) = \nabla^2 f(x_2) - \nabla^2 f(x_1)$	$\nabla^4 f(x_0) = \nabla^3 f(x_1) - \nabla^3 f(x_0)$
x_3	$f(x_3)$	$\nabla f(x_3) = f(x_4) - f(x_3)$	$\nabla^2 f(x_2) = \nabla f(x_3) - \nabla f(x_2)$		
x_4	$f(x_4)$				

Table 3.3: Backward Difference Table

The backward differences can be written in a tabular form as in Table 3.3

■ **Example 3.11** Construct the backward difference table for the data

x	-1	0	1	2
$f(x)$	-8	3	1	12

■

Solution: We have the following difference table:

3.2.2 Relation Between finite difference operator

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
-1	-8			
		$3-(-8)=11$		
0	3		$-2-11=-13$	
		$1-3=-2$		$13+13=26$
1	1		$11+2=13$	
		$12-1=11$		
2	12			

Table 3.4: Backward Difference Table

Definition 3.2.4 Central difference operator δ: The central difference operator is defined as

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$$

Definition 3.2.5 Average (Mean) operator μ: The central difference operator is defined as

$$\mu f(x) = \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}$$

Prove that: $(i)\quad \Delta = E-1 \quad (ii)\quad \Delta-\nabla = \nabla\Delta \quad (iii)\quad (1+\Delta)(1-\nabla)=1$

$(iv)\quad \mu = \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right] \quad (v)\quad \delta = \nabla(1-\nabla)^{-1/2} \quad (vi)\quad \mu = \left[1+\frac{\delta^2}{4}\right]^{1/2}$

$(vii)\quad E = e^{hD}$, where $D = \frac{d}{dx}$.

Proof: (i) We know that $Ef(x) = f(x+h)$. Therefore,

$$\Delta f(x) = f(x+h) - f(x) \implies \Delta f(x) = Ef(x) - f(x) \implies \Delta = E-1 \tag{3.1}$$

(ii) L.H.S

$$\begin{aligned}(\Delta-\nabla)f(x) &= \Delta f(x) - \nabla f(x)\\ &= \{f(x+h)-f(x)\} - \{f(x)-f(x-h)\}\\ &= f(x+h)-2f(x)+f(x-h)\end{aligned}$$

R.H.S

$$\begin{aligned}(\nabla\Delta)f(x) &= \nabla\{\Delta f(x)\}\\ &= \nabla\{f(x+h)-f(x)\}\\ &= \{\nabla f(x+h) - \nabla f(x)\}\\ &= \{(f(x+h)-f(x)) - (f(x)-f(x-h)\}\\ &= f(x+h)-2f(x)+f(x-h)\end{aligned}$$

Thus $L.H.S. = R.H.S.$

(iii) We know that $\nabla f(x) = f(x)-f(x-h) \implies \nabla f(x) = f(x) - E^{-1}f(x) \implies \nabla = 1-E^{-1} \implies 1-\nabla = E^{-1}$ and $\Delta = E-1 \implies 1+\Delta = E$. Therefore,

$$(1+\Delta)(1-\nabla) = (E)(E^{-1}) = 1$$

(iv) We know that

$$\begin{aligned}
\mu f(x) & \qquad\qquad (3.2)\\
&= \frac{1}{2}\left[f(x+\frac{h}{2})+f(x-\frac{h}{2})\right]\\
&= \frac{1}{2}\left[E^{1/2}f(x)+E^{-1/2}f(x)\right]\\
\mu f(x) &= \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right]f(x).
\end{aligned}$$

(v) We know that $1-\nabla = E^{-1}$Therefore,

$$\begin{aligned}
R.H.S. &= \nabla(1-\nabla)^{-1/2}f(x)\\
&= \nabla(E^{-1})^{-1/2}f(x)\\
&= \nabla\left\{(E^{1/2})f(x)\right\} = \nabla f(x+\frac{h}{2})\\
&= f(x+\frac{h}{2})-f(x+\frac{h}{2}-h)\\
&= f(x+\frac{h}{2})-f(x+\frac{-h}{2})\\
&= \delta f(x) = L.H.S.
\end{aligned}$$

(vi) Try Yourself.

(v) We know that the Taylor's series

$$\begin{aligned}
Ef(x) = f(x+h) & \\
&= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\ldots\\
&= f(x)+hDf(x)+\frac{h^2}{2}D^2f(x)+\ldots\\
&= \left[1+hD+\frac{h^2}{2}D^2+\ldots\right]f(x)\\
Ef(x) &= e^{hD}f(x).
\end{aligned}$$

3.3 Newton's Forward Difference Interpolation Formula

Newton's Forward Interpolation formula: Let $x_0, x_1, x_2, \ldots, x_n$ be the equally spaced data and h be the step length in the given data. In terms of the divided differences, we have the interpolation formula as

$$f(x) = f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+\ldots$$

Using the relations for the divided differences

$$f[x_0,x_1,\ldots,x_n] = \frac{1}{n!h^n}\Delta^n f(x_0)$$

we get

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f(x_0)}{3!h^3} + \ldots + (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})\frac{\Delta^n f(x_0)}{n!h^n}.$$

This relation is called the *Newton's forward difference interpolation* formula.

■ **Example 3.12** For the data construct the forward difference formula. Hence, find $f(0.5)$. ■

x	-2	-1	0	1	2	3
$f(x)$	15	5	1	3	11	25

Solution: We have the following difference table: From the table, we conclude that the data

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-2	15					
		-10				
-1	5		6			
		-4		0		
0	1		6		0	
		2		0		0
1	3		6		0	
		8		0		
2	11		6			
		14				
3	25					

Table 3.5: Forward Difference Table

represents a quadratic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f(x_0)}{3!h^3} + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\frac{\Delta^4 f(x_0)}{4!h^4} + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\frac{\Delta^5 f(x_0)}{5!h^5}.$$

By putting the required values from the table we have,

$$f(x) = 15 + (x+2)\frac{(-10)}{1!.1} + (x+2)(x+1)\frac{6}{2!1^2} + (x+2)(x+1)(x-0)\frac{0}{3!1^3} + (x+2)(x+1)(x-0)(x-1)\frac{0}{4!1^4} + (x+2)(x+1)(x-0)(x-1)(x-2)\frac{0}{5!1^5}.$$

$$f(x) = 15 + (x+2)(-10) + (x+2)(x+1)(3) = 15 - 10x - 20 + 3x^2 + 9x + 6 = 3x^2 - x + 1.$$

We obtain $f(0.5) = 3(0.5)^2 - 0.5 + 1 = 0.75 - 0.5 + 1 = 1.25$.

■ **Example 3.13** A third degree polynomial passes through the points $(0,-1),(1,1),(2,1) and (3,-2)$. Determine this polynomial using Newton's forward interpolation formula. Hence, find the value at 1.5. ■

Solution: We have the following difference table: From the table, we conclude that the data

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-1			
		2		
1	1		-2	
		0		-1
2	1		-3	
		-3		
3	-2			

Table 3.6: Forward Difference Table

represents a cubic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!h} + (x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!h^3}.$$

By putting the required values from the table we have,

$$f(x) = (-1) + (x-0)\frac{2}{1!.1} + (x-0)(x-1)\frac{-2}{2!1^2} + (x-0)(x-1)(x-2)\frac{-1}{3!1^3}.$$

$$f(x) = -1 + 2x - x(x-1) - \frac{1}{6}x(x-1)(x-2) = -1 + 2x - x^2 + x - \frac{1}{6}(x^3 - 3x^2 + 2x)$$

$$f(x) = -1 + (2 - 2/6 + 1)x - (1 - 3/6)x^2 - 1/6x^3 = -1 + (8/3)x - (1/2)x^2 - (1/6)x^3$$

We obtain $f(1.5) = -1 + (8/3)(1.5) - (1/2)(1.5)^2 - (1/6)(1.5)^3 = 1.3125$.

3.4 Newton's Backward Difference Interpolation Formula

Newton's Backward Interpolation formula Let $x_0, x_1, x_2, \ldots, x_n$ be the equally spaced data and h be the step length in the given data. Again, we use the Newton's divided difference interpolation polynomial to derive the Newton's backward difference interpolation formula. Since, the divided differences are symmetric with respect to their arguments, we write the arguments of the divided differences in the order $x_n, x_{n-1}, \ldots, x_1, x_0$. The Newton's divided difference interpolation polynomial can be written as

$$f(x) = f(x_n) + (x-x_n)f[x_n,x_{n-1}] + (x-x_n)(x-x_{n-1})f[x_n,x_{n-1},x_{n-2}] + \ldots + (x-x_n)(x-x_{n-1})\ldots(x-x_1)f[x_n,x_{n-1},x_{n-2},\ldots,x_1,x_0]$$

Since, the divided differences are symmetric with respect to their arguments, we have

$$f[x_n,x_{n-1},\ldots,x_0] = f[x_0,x_1,\ldots,x_n] = \frac{1}{n!h^n}\nabla^n f(x_n)$$

Thus we obtain the *Newton's backward difference interpolation* formula as

$$f(x) = f(x_n) + (x-x_n)\frac{\nabla f(x_n)}{1!h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f(x_n)}{2!h^2} + (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\nabla^3 f(x_n)}{3!h^3} + \ldots + (x-x_n)(x-x_{n-1})(x-x_{n-2})\ldots(x-x_1)\frac{\nabla^n f(x_n)}{n!h^n}.$$

■ **Example 3.14** Using Newton's backward difference interpolation, interpolate at $x = 1.0$ from the following data. ■

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Solution: We have the following difference table: From the table, We have $h = 0.2$. The Newton's

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0.1	-1.699					
		0.626				
0.3	-1.073		0.072			
		0.698		0.048		
0.5	-0.375		0.120		0	
		0.818		0.048		0
0.7	0.443		0.168		0	
		0.986		0.048		
0.9	1.429		0.216			
		1.202				
1.1	2.631					

Table 3.7: Backward Difference Table

backward difference formula is given by

$$f(x) = f(x_n) + (x-x_n)\frac{\nabla f(x_n)}{1!h} + (x-x_n)(x-x_{n-1})\frac{\nabla^2 f(x_n)}{2!h^2} + (x-x_n)(x-x_{n-1})(x-x_{n-2})\frac{\nabla^3 f(x_n)}{3!h^3}.$$

By putting the required values from the table we have,

$$f(x) = 2.631 + (x-1.1)\frac{1.202}{1!(0.2)} + (x-1.1)(x-0.9)\frac{0.216}{2!(0.2)^2} + (x-1.1)(x-0.9)(x-0.7)\frac{0.048}{3!(0.2)^3}.$$

$$f(x) = 2.631 + 6.01(x-1.1) + 2.7(x-1.1)(x-0.9) + (x-1.1)(x-0.9)(x-0.7).$$

Since, we have not been asked to find the interpolation polynomial, we may not simplify this expression. At $x = 1.0$, we obtain
$f(1.0) = 2.631 + 6.01(1.0-1.1) + 2.7(1.0-1.1)(1.0-0.9) + (1.0-1.1)(1.0-0.9)(1.0-0.7)$
$= 2.631 + 6.01(-0.1) + 2.7(-0.1)(0.1) + (-0.1)(0.1)(-0.3) = 2.004.$

Quiz

Question 1: For the following data, calculate the differences and obtain the Newton's forward and backward difference interpolation polynomials. Are these polynomials different? Interpolate at $x = 0.25$ and $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Question 2: Give the relation between the divided differences and forward or backward differences.
Question 3: Can we decide the degree of the polynomial that a data represents by writing the forward or backward difference tables?

Definition 3.4.1 Divided Difference Let the $(x_i, f(x_i)), i = 0, 1, 2, \ldots, n$ be given unequal spaced data. We define the divided differences as follows:
First divided difference: Consider any two consecutive data values $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$. Then, we define the first divided difference as

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

In particular,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2}, \ldots$$

Second divided difference: Consider any three consecutive data values $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2}))$. Then, we define the second divided difference as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

In particular,

$$f[x_0, x_1, x_2] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}, f[x_1, x_2, x_3] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1}, \ldots$$

■ **Example 3.15** Find the second divided difference of $f(x) = 1/x$, using the points a, b, c. ■

Solution: We have

$$f[a,b] = \frac{f(b) - f(a)}{b - a} = \frac{(1/b) - (1/a)}{b - a} = \frac{(a-b)/ab}{b - a} = -\frac{1}{ab}$$

$$f[b,c] = \frac{f(c) - f(b)}{c - b} = \frac{(1/c) - (1/b)}{c - b} = \frac{(b-c)/bc}{c - b} = -\frac{1}{bc}$$

$$f[a,b,c] = \frac{f[b,c] - f[a,b]}{c - a} = \frac{(-1/bc) - (-1/ab)}{c - a} = \frac{1}{abc}$$

■ **Example 3.16** Obtain the divided difference table for the data

x	-1	0	2	3
f(x)	-8	3	1	12

■

Solution: We have the following divided difference table:

x	$f(x)$	$first \quad D.D.$	$Second \quad D.D.$	$Third \quad D.D.$
-1	-8			
		$\frac{3-(-8)}{0-(-1)} = 11$		
0	3		$\frac{-1-11}{2-(-1)} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4-(-4)}{3-(-1)} = 2$
2	1		$\frac{11-(-1)}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

Table 3.8: Divided Difference Table

3.5 Newton Divided difference Interpolation

Definition 3.5.1 Newton Divided Difference Interpolation Let the $(x_i, f(x_i)), i = 0, 1, 2, \ldots, n$ be given unequal spaced data. We define the Newton divided difference interpolation formula as follows:

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1})f[x_0, x_1, x_2, \ldots, x_n].$$

■ **Example 3.17** Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	-4	-1	0	2	5
f(x)	1245	33	5	9	1335

■

Solution: We form the divided difference table for the given data. The Newton's divided difference formula gives

$$\begin{aligned}
f(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
&+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3, x_4]. \\
&= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)x(-14) \\
&+ (x + 4)(x + 1)x(x - 2)(3). \\
&= 1245 - 404x - 1616 + (x^2 + 5x + 4)(94) + (x^3 + 5x^2 + 4x)(-14) + (x^4 + 3x^3 - 6x^2 - 8x)(3). \\
&= 3x^4 - 5x^3 + 6x^2 - 14x + 5.
\end{aligned}$$

x	$f(x)$	*first D.D.*	*Second D.D.*	*Third D.D.*	*Fourth D.D.*
-4	1245	$\frac{33-1245}{-1-(-4)}=-404$			
-1	33	$\frac{5-33}{0-(-1)}=-28$	$\frac{-28-(-404)}{0-(-4)}=94$	$\frac{10-(94)}{2-(-4)}=-14$	
0	5	$\frac{9-5}{2-0}=2$	$\frac{2-(-28)}{2-(-1)}=10$	$\frac{88-10}{5-(-1)}=13$	$\frac{13-(-14)}{5-(-4)}=3$
2	9	$\frac{1335-9}{5-2}=442$	$\frac{442-(2)}{5-0}=88$		
5	1335				

Table 3.9: Divided Difference Table

■ **Example 3.18** Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	1	3	4	5	7	10
$f(x)$	3	31	69	131	351	1011

Hence, interpolate at $x = 3.5$ and $x = 8.0$. Also find, $f'(3)$ and $f''(1.5)$. ■

Solution: We form the divided difference table for the given data.

x	$f(x)$	*first D.D.*	*Second D.D.*	*Third D.D.*	*Fourth D.D.*	*Fifth D.D.*
1	3	14				
3	31	38	8	1		
4	69	62	12	1	0	0
5	131	110	16	1	0	
7	351	220	22			
10	1011					

Table 3.10: Divided Difference Table

The Newton's divided difference formula gives

$$\begin{aligned} f(x) &= f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] \\ &+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0,x_1,x_2,x_3,x_4] \\ &+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)f[x_0,x_1,x_2,x_3,x_4,x_5]. \\ &= 3+(x-1)(14)+(x-1)(x-3)(8)+(x-1)(x-3)(x-4)(1). \\ &= 3+14x-14+8x^2-32x+24+x^3-8x^2+19x-12 = x^3+x+1. \end{aligned}$$

Hence $f(3.5) = P_3(3.5) = (3.5)3+3.5+1 = 47.375$, and $f(8.0) = P_3(8.0) = (8.0)3+8.0+1 = 521.0$.
Now, $P_3'(x) = 3x^2+1$, and $P_3''(x) = 6x$.
Therefore, $f'(3) = P_(' 3) = 3(9)+1 = 28, f''(1.5) = P_3''(1.5) = 6(1.5) = 9$.

3.6 Lagrange's Interpolation formula:

Let the data

x	x_0	x_1	x_2	...	x_n
f(x)	$f(x_1)$	$f(x_2)$	$f(x_2)$	...	$f(x_n)$

be given at distinct unevenly spaced points or non-uniform points $x_0, x_1, \dots, x_n$. This data may also be given at evenly spaced points. For this data, we can fit a unique polynomial of degree $\leq n$. Since the interpolating polynomial must use all the ordinates $f(x_0), f(x_1), \dots f(x_n)$, it can be written as a linear combination of these ordinates. That is, we can write the polynomial as

$$P_n(x) = l_0(x)f(x_0)+l_1(x)f(x_1)+\dots+l_n(x)f(x_n)$$

where

$$l_i(x) = \frac{(x-x_0),(x-x_1),(x-x_2),\dots,(x-x_{i-1}),(x-x_{i+1}),\dots,(x-x_n)}{(x_i-x_0),(x_i-x_1),(x_i-x_2),\dots,(x_i-x_{i-1}),(x_i-x_{i+1}),\dots,(x_i-x_n)}$$

■ **Example 3.19** Use Lagrange's formula, to find the quadratic polynomial that takes the values

x	0	1	3
f(x)	0	1	0

■

Solution: Since $f(x_0)$ and $f(x_2)$ are zero, we need to compute $l_1(x)$ only. We have

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x^2-3x)$$

The Lagrange quadratic polynomial is given by

$$P_2(x) = f(x) = l_0f(x_0)+l_1f(x_1)+l_2f(x_2) = 0+-\frac{1}{2}(x^2-3x)(1)+0 = \frac{1}{2}(3x-x^2).$$

■ **Example 3.20** Construct the Lagrange interpolation polynomial for the data

x	-1	1	4	7
$f(x)$	-2	0	63	342

Hence, interpolate at $x = 5$. ■

Solution: Since $f(x_1)$ is zero, we need to compute $l_0(x)$, $l_2(x)$, $l_3(x)$ only. We have

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-4)(x-7)}{(-1-1)(-1-4)(-1-7)} = -\frac{1}{80}(x^3-12x^2+39x-28).$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-1)(x-7)}{(4+1)(4-1)(4-7)} = -\frac{1}{45}(x^3-7x^2-x+7).$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)} = \frac{1}{144}(x^3-4x^2-x+4).$$

The Lagrange quadratic polynomial is given by

$$\begin{aligned} f(x) &= l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) + l_3 f(x_3) \\ &= -\frac{1}{80}(x^3-12x^2+39x-28)(-2) + 0 - \frac{1}{45}(x^3-7x^2-x+7)(63) + \frac{1}{144}(x^3-4x^2-x+4)(342) \\ &= \left(\frac{1}{40} - \frac{7}{5} + \frac{171}{72}\right)x^3 + \left(-\frac{3}{10} + \frac{49}{5} - \frac{171}{18}\right)x^2 + \left(\frac{39}{40} + \frac{7}{5} - \frac{171}{72}\right)x + \left(-\frac{7}{10} - \frac{49}{5} + \frac{171}{8}\right) \\ &= x^3 - 1. \end{aligned}$$

Hence, $f(5) = P_3(5) = 5^3 - 1 = 124$.

■ **Example 3.21** Given that $f(0) = 1, f(1) = 3, f(3) = 55$, find the unique polynomial of degree 2 or less, which fits the given data. ■

Solution: We have $x_0 = 0, f(x_0) = 1, x_1 = 1, f(x_1) = 3, x_2 = 3, f(x_2) = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2-4x+3).$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{1}{2}(3x-x^2).$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x^2-x).$$

The Lagrange quadratic polynomial is given by

$$\begin{aligned} P_2(x) = f(x) &= l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) \\ &= \frac{1}{3}(x^2-4x+3)(1) + \frac{1}{2}(3x-x^2)(3) + \frac{1}{6}(x^2-x)(55) \\ &= 8x^2 - 6x + 1. \end{aligned}$$

Quiz

Question 1: Using the data $sin(0.1) = 0.09983$ and $sin(0.2) = 0.19867$, find an approximate value of $sin(0.15)$ by Lagrange interpolation.
Question 2: Give two uses of interpolating polynomials.
Question 3: Write the property satisfied by Lagrange fundamental polynomials $l_i(x)$.

3.7 Numerical Integration

Numerical Integration using Trapezoidal rule:

This rule is also called the trapezeum rule. Let the curve $y = f(x), a \leq x \leq b$, be approximated by the line joining the points $P(a, f(a)), Q(b, f(b))$ on the curve. Let the interval $[a, b]$ be subdivided into N equal parts of length h. That is, $h = (b-a)/N$. The nodal points are given by $a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_N = x_0 + Nh = b$. The *Trapezoidal rule* is defined as

$$I = \int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_N) + 2\left\{f(x_1) + f(x_2) + \ldots + f(x_{N-1})\right\}\right].$$

Remarks: The trapezium rule produces exact results for polynomials of degree ≤ 1.

■ **Example 3.22** Using the trapezium rule, evaluate the integral $I = \int_0^1 \frac{dx}{x^2+6x+10}$ with 2 and 4 subintervals. Compare with the exact solution. Comment on the magnitudes of the errors obtained ■

Solution: With $N = 2$ and 4, we have the following step lengths and nodal points.

$$N = 2, \qquad h = \frac{b-a}{N} = \frac{1}{2}. \text{ The nodes are } 0, 0.50, 1.0.$$

We have the following tables of values.

x	0	0.50	1.0
$f(x)$	0.1	0.07547	0.05882

Now, we compute the value of the integral.

$$\begin{aligned} I_1 = \int_0 \frac{dx}{x^2+6x+10} &= \frac{h}{2}\left[f(0) + f(1.0) + 2\left\{f(0.50)\right\}\right] \\ &= 0.50\left[0.1 + 0.05882 + 2\left\{0.07547\right\}\right]. \\ &= 0.07744. \end{aligned}$$

$$N = 4, \qquad h = \frac{b-a}{N} = \frac{1}{4}. \; h = 0.25, \text{ The nodes are } 0.0, 0.25, 0.5, 0.75, 1.0.$$

We have the following tables of values.

x	0.0	0.25	0.50	0.75	1.0
$f(x)$	0.1	0.08649	0.07547	0.06639	0.05882

Now, we compute the value of the integral.

$$\begin{aligned} I_2 &= \int_0^1 \frac{dx}{x^2+6x+10} \\ &= \frac{h}{2}\left[f(0.0) + f(1.0) + 2\left\{f(0.25) + f(0.50) + f(0.75)\right\}\right] \\ &= 0.125\left[0.1 + 0.05882 + 2\left\{0.08649 + 0.07547 + 0.06639\right\}\right]. \\ &= 0.07694. \end{aligned}$$

The exact value of the integral is

$$I = \int_0^1 \frac{dx}{x^2+6x+10} = \int_0^1 \frac{dx}{(x+3)^2+1} = \left[\tan^{-1}(x+3)\right]_0^1 = \left[\tan^{-1}(4) - \tan^{-1}(3)\right] = 0.07677$$

The errors in the solutions are the following:

$$|Exact - I_1| = |0.07677 - 0.07744| = 0.00067.$$

$$|Exact - I_2| = |0.07677 - 0.07694| = 0.00017.$$

We find that $|Error \quad in \quad I_2| = \frac{1}{4}|Error \quad in \quad I_1|$.

■ **Example 3.23** Evaluate $I = \int_1^2 \frac{dx}{5+3x}$ with 4 and 8 subintervals using the trapezium rule. Compare with the exact solution and find the absolute errors in the solutions. Comment on the magnitudes of the errors obtained. Find the bound on the errors. ■

Solution: With $N = 4$ and 8, we have the following step lengths and nodal points.

$$N = 4, \qquad h = \frac{b-a}{N} = \frac{1}{4}. \text{ The nodes are } 1, 1.25, 1.5, 1.75, 2.0.$$

We have the following tables of values.

x	1	1.25	1.50	1.75	2.00
$f(x)$	0.125	0.11429	0.10526	0.09756	0.09091

Now, we compute the value of the integral.

$$\begin{aligned} I_1 = \int_1^2 \frac{dx}{5+3x} &= \frac{h}{2}[f(1.0) + f(2.0) + 2\{f(1.25) + f(1.50) + f(1.75)\}] \\ &= 0.125\,[0.125 + 0.09091 + 2\{0.11429 + 0.10526 + 0.09756\}]. \\ &= 0.10627. \end{aligned}$$

$$N = 8, \qquad h = \frac{b-a}{N} = \frac{1}{8}. \text{ The nodes are } 1, 1.125, 1.25, 1.375, 1.5, 1.675, 1.75, 1.875, 2.0.$$

We have the following tables of values.

x	1	1.125	1.25	1.375	1.50	1.675	1.75	1.875	2.00
$f(x)$	0.125	0.11940	0.11429	0.10959	0.10526	0.10127	0.09756	0.09412	0.09091

Now, we compute the value of the integral.

$$\begin{aligned} I_2 &= \int_1^2 \frac{dx}{5+3x} \\ &= \frac{h}{2}[f(1.0) + f(2.0) + 2\{f(1.125) + f(1.25) + f(1.375) + f(1.50) + f(1.675) + f(1.75) + f(1.875)\}] \\ &= 0.0625\,[0.125 + 0.09091 + 2\{0.11940 + 0.11429 + 0.10959 + 0.10526 + 0.10127 + 0.09756 + 0.09412\}]. \\ &= 0.10618. \end{aligned}$$

The exact value of the integral is

$$I = \int_1^2 \frac{dx}{5+3x} = \frac{1}{3}[\ln(5+3x)]_1^2 = \frac{1}{3}[\ln(11) - \ln(8)] = 0.10615$$

The errors in the solutions are the following:

$$|Exact - I_1| = |0.10615 - 0.10627| = 0.00012.$$

$$|Exact - I_2| = |0.10615 - 0.10618| = 0.00003.$$

We find that $|Error \quad in \quad I_2| = \frac{1}{4}|Error \quad in \quad I_1|$.

Quiz

Question 1: Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$, using the trapezium rule with 2, 4 and 8 equal subintervals. Using the exact solution, find the absolute errors.
Question 2: What is the restriction in the number of nodal points, required for using the trapezium rule for integrating $I = \int_a^b f(x)dx$?
Question 3: What is the geometric representation of the trapezium rule for integrating $I = \int_a^b f(x)dx$?

3.8 Numerical Integration using Simpson 1/3 rule or Simpson 3/8 rule:

Simpson 1/3 rule: We subdivide the given interval [a, b] into even number of subintervals of equal length h. That is, we obtain an odd number of nodal points. We take the even number of intervals as $2N$. The step length is given by $h = (b-a)/(2N)$. The nodal points are given by $a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_{2N} = x_0 + 2Nh = b$. Then, Simpson 1/3 rule is defined as

$$I = \int_a^b f(x)dx =$$
$$\frac{h}{3}[f(x_0) + f(x_{2N}) + 4\{f(x_1) + f(x_3) + ... + f(x_{2N-1})\} + 2\{f(x_2) + f(x_4) + ... + f(x_{2N-2})\}].$$

■ **Example 3.24** Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$, using the Simpson's 1/3 rule with 2, 4 and 8 equal subintervals. Using the exact solution, find the absolute errors. ■

Solution: With $n = 2N = 2$ or $N = 1$ we have the following step lengths and nodal points.

$$\text{For } N = 1, \quad h = \frac{b-a}{2N} = \frac{1-0}{2} = 0.5, \text{ The nodes are } 0, 0.5, 1.0.$$

We have the following tables of values.

x	0	0.5	1.0
$f(x)$	1	0.666667	0.5

Now, we compute the value of the integral.

$$\begin{aligned} I_1 &= \int_0^1 \frac{dx}{1+x} \\ &= \frac{h}{3}[f(0) + f(1.0) + 4\{f(0.5)\}] \\ &= \frac{0.5}{3}[1.0 + 0.5 + 4\{0.666667\}]. \\ &= 0.674444. \end{aligned}$$

Again, with $n = 2N = 4$ or $N = 2$ we have the following step lengths and nodal points.

For $N=2$, $h=\frac{b-a}{2N}=\frac{1-0}{4}=0.25$, The nodes are $0, 0.25, 0.5, 0.75, 1.0$.

We have the following tables of values.

x	0	0.25	0.5	0.75	1.0
$f(x)$	1	0.8	0.666667	0.571429	0.5

Now, we compute the value of the integral.

$$\begin{aligned} I_2 &= \int_0^1 \frac{dx}{1+x} \\ &= \frac{h}{3}[f(0)+f(1.0)+4\{f(0.25)+f(0.75)\}+2\{f(0.5)\}] \\ &= \frac{0.5}{3}[1.0+0.5+4\{0.8+0.571429\}+2(0.666667)]. \\ &= 0.693254. \end{aligned}$$

Finally, with $n=2N=8$ or $N=4$ we have the following step lengths and nodal points. For $N=4$, $h=\frac{b-a}{2N}=\frac{1-0}{8}=0.125$,

The nodes are $0, 0.125, 0.250, 0.375, 0.5, 0.625, 0.75, 0.875, 1.0$.

We have the following tables of values.

x	0	0.125	0.250	0.375	0.500	0.675	0.750	0.875	1.0
$f(x)$	1	0.888889	0.8	0.727273	0.666667	0.615385	0.571429	0.533333	0.5

Now, we compute the value of the integral.

$$\begin{aligned} I_3 &= \int_0^1 \frac{dx}{1+x} \\ &= \frac{h}{3}[f(0)+f(1.0)+4\{f(0.125)+f(0.375)+f(0.675)+f(0.875)\}+2\{f(0.25)+f(0.5)+f(0.75)\}] \\ &= \frac{0.5}{3}[1.0+0.5+4\{0.888889+0.727273+0.615385+0.533333\}+2\{0.8+0.666667+0.571429\}]. \\ &= 0.693155. \end{aligned}$$

The exact value of the integral is $I=\ln 2=0.693147$. The errors in the solutions are the following:

$$|Exact - I_1| = |0.693147 - 0.694444| = 0.001297.$$

$$|Exact - I_2| = |0.693147 - 0.693254| = 0.000107.$$

$$|Exact - I_3| = |0.693147 - 0.693155| = 0.000008.$$

■ **Example 3.25** The velocity of a particle which starts from rest is given by the following table.

$t(sec)$	0	2	4	6	8	10	12	14	16	18	20
$v(ft/sec)$	0	16	29	40	46	51	32	18	8	3	0

Evaluate using Simpson's $1/3$ rule, the total distance traveled in 20 seconds. ■

Solution: From the definition, we have

$$v = \frac{ds}{dt} \text{ or } s = \int v dt$$

Starting from rest, the distance traveled in 20 seconds is

$$s = \int_0^{20} v dt$$

The step length is $h = 2$. Using the Simpson's rule, we obtain

$$\begin{aligned} s &= \int_0^{20} v dt \\ &= \frac{h}{3}[f(0) + f(20) + 4\{f(2) + f(6) + f(10) + f(14) + f(18)\} + 2\{f(4) + f(8) + f(12) + f(16)\}] \\ &= \frac{2}{3}[0 + 0 + 4\{16 + 40 + 51 + 18 + 3\} + 2\{29 + 46 + 32 + 8\}]. \\ &= 494.667 \quad feet. \end{aligned}$$

3.9 Simpson $3/8$ rule:

In Simpson's $3/8$ rule, the number of subintervals is $n = 3N$. Hence, we have

$$h = \frac{b-a}{3N}.$$

and *Simpson* $3/8$ *rule* is defined as

$$I = \int_a^b f(x)dx = \frac{3h}{8}[f(x_0) + f(x_{3N}) + 2\{f(x_3) + f(x_6) + \ldots + f(x_{3N-3})\} + 3\{f(x_1) + f(x_2) + \ldots + f(x_{2N-2}) + f(x_{2N-1})\}].$$

■ **Example 3.26** Evaluate $I = \int_1^2 \frac{dx}{5+3x}$ with 3 and 6 subintervals using Simpson's $3/8$ rule. Compare with the exact solution. ■

Solution: With $N = 3$ and 6, we have the following step lengths and nodal points.

$$N = 3, \qquad h = \frac{b-a}{N} = \frac{1}{3}. \text{ The nodes are } 1, 4/3, 5/3, 2.0.$$

We have the following tables of values.

x	1	4/3	5/3	2.00
$f(x)$	0.125	0.11111	0.10000	0.09091

Now, we compute the value of the integral.

$$\begin{aligned} I_1 = \int_1^2 \frac{dx}{5+3x} &= \frac{3h}{8}[f(1.0) + f(2.0) + 3\{f(4/3) + f(5/3)\}] \\ &= 0.125[0.125 + 0.09091 + 3\{0.11111 + 0.10000\}]. \\ &= 0.10616. \end{aligned}$$

$$N = 6, \qquad h = \frac{b-a}{N} = \frac{1}{6}.\ \text{The nodes are } 1, 7/6, 8/6, 9/6, 10/6, 11/6, 2.0.$$

We have the following tables of values.

x	1	7/6	8/6	9/6	10/6	11/6	2.00
$f(x)$	0.125	0.11765	0.11111	0.10526	0.10000	0.09524	0.09091

Now, we compute the value of the integral.

$$\begin{aligned} I_2 &= \int_1^2 \frac{dx}{5+3x} \\ &= \frac{3h}{8}\left[f(1.0)+f(2.0)+2\{f(9/6)\}+3\{f(7/6)+f(8/6)+f(10/6)+f(11/6)\}\right] \\ &= \frac{1}{16}\left[0.125+0.09091+2\{0.10526\}+3\{0.11765+0.11111+0.10000+0.09524\}\right]. \\ &= 0.10615. \end{aligned}$$

The exact value of the integral is

$$I = \int_1^2 \frac{dx}{5+3x} = \frac{1}{3}\left[\ln(5+3x)\right]_1^2 = \frac{1}{3}\left[\ln(11) - \ln(8)\right] = 0.10615$$

The errors in the solutions are the following:

$$|Exact - I_1| = |0.10615 - 0.10616| = 0.00001.$$

$$|Exact - I_2| = |0.10615 - 0.10615| = 0.00000.$$

The magnitude of the error for $N = 3$ is 0.00001 and for $N = 6$ the result is correct to all places.

Remarks: The Simpson 1/3 rule and Simpson 3/8 rule produces exact results for polynomials of degree ≤ 3.

Quiz

Question 1: Find the approximate value of $I = \int_1^2 \frac{dx}{5+3x}$, using the Simpson 1/3 rule with 4 and 8 equal subintervals. Using the exact solution, find the absolute errors.
Question 2: What are the disadvantages of the Simpson's 3/8 rule compared with the Simpson's 1/3 rule?
Question 3: What is the restriction in the number of nodal points, required for using the Simpson's 3/8 rule for integrating $I = \int_a^b f(x)dx$?

3.10 Solution of ordinary differential equations by Taylor's Series Method:

The Taylor's series is defined as

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \ldots$$

■ **Example 3.27** Consider the initial value problem $y' = x(y+1), y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$ using Taylor series method of order two and fourth. If the exact solution is $y = -1 + 2e^{x^2/2}$, find the magnitudes of the actual errors for $y(0.2)$. ■

Solution: We have $y' = f(x,y) = x(y+1), \quad x_0 = 0, \quad y_0 = 1$ and $h = 0.1$
(i) Taylor series second order method.

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i''$$

We have $y'' = xy' + y + 1$. With $x_0 = 0, y_0 = 1$, we get $y'(0) = 0, y''(0) = x_0y_0' + y_0 + 1 = 0 + 1 + 1 = 2$.

$$y(0.1) = y_1 = y_0 + (0.1)y_0' + \frac{(0.1)^2}{2!}y_0'' = 1 + 0 + (0.005)2 = 1.01$$

With $x_1 = 0.1, y_1 = 1.01$, we get $y_1' = 0.1(1.01+1) = 0.201$. and $y_1'' = x_1y_1' + y_1 + 1 = (0.1)(0.201) + 1.01 + 1 = 2.0301$.

$$y(0.2) = y_2 = y_1 + (0.1)y_1' + \frac{(0.1)^2}{2!}y_1'' = 1.01 + 0.1(0.201) + 0.005(2.0301) = 1.04025.$$

(ii) Taylor series fourth order method.

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{iv}$$

We have $y'' = xy' + y + 1, y''' = xy'' + 2y', y^{iv} = xy''' + 3y''$.
With $x_0 = 0, y_0 = 1$, we get

$$y_0' = 0, y_0'' = 2, y_0''' = x_0y_0'' + 2y_0' = 0, y_0^{iv} = x_0y_0''' + 3y_0'' = 0 + 3(2) = 6.$$

$$y_1 = y_0 + (0.1)y_0' + \frac{(0.1)^2}{2!}y_0'' + \frac{(0.1)^3}{3!}y_0''' + \frac{(0.1)^4}{4!}y_0^{iv}$$

$$= 1 + 0 + 0.005(2) + 0 + \frac{0.0001}{24}(6) = 1.010025.$$

With $x_1 = 0.1, y_1 = 1.010025$, we get

$$y_1' = 0.1(1.010025 + 1) = 0.201003.$$

$$y_1'' = x_1y_1'' + y_1 + 1 = (0.1)(0.201003) + 1.010025 + 1 = 2.030125.$$

$$y_1''' = x_1y_1'' + 2y_1' = 0.1(2.030125) + 2(0.201003) = 0.605019,$$

$$y_1^{iv} = x_1y_1'' + 3y_1'' = 0.1(0.605019) + 3(2.030125) = 6.150877.$$

$$y_2 = y_1 + (0.1)y_1' + \frac{(0.1)^2}{2!}y_1'' + \frac{(0.1)^3}{3!}y_1''' + \frac{(0.1)^4}{4!}y_1^{iv}$$

$$= 1.010025 + 0.1(0.201003) + 0.005(2.030125) + + \frac{0.001}{6}(0.605019) + \frac{0.0001}{24}(6.150877) = 1.040402.$$

The exact value is $y(0.1) = 1.010025, y(0.2) = 1.040403$.
The magnitudes of the actual errors at $x = 0.2$ are

Taylor series method of second order: $|1.04025 - 1.040403| = 0.000152$.

Taylor series method of fourth order: $|1.040402 - 1.040403| = 0.000001$.

3.11 Solution of ordinary differential equations by Euler's method:

Consider a first order initial value problem defined as

$$y' = f(x,y), \qquad y(x_0) = y_0$$

The Euler's method is defined as

$$y_{n+1} = y_n + hf(x_n, y_n).$$

■ **Example 3.28** Solve the initial value problem $yy' = x, \quad y(0) = 1$, using the Euler method in $0 \le x \le 0.8$, with $h = 0.2$ and $h = 0.1$. Compare the results with the exact solution at $x = 0.8$.. ■

Solution: We have $y' = f(x,y) = \frac{x}{y}$. The Euler's method gives

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + h\left(\frac{x_n}{y_n}\right).$$

Here $h = 0.2, x_0 = 0, y_0 = 1$. Now we have

$$y_1 = y(x_1) = y(0.2) = y_0 + h\left(\frac{x_0}{y_0}\right) = 1 + (0.2)(0) = 1.0$$

$$y_2 = y(x_2) = y(0.4) = y_1 + h\left(\frac{x_1}{y_1}\right) = 1 + (0.2)\left(\frac{0.2}{1.0}\right) = 1.04$$

$$y_3 = y(x_3) = y(0.6) = y_2 + h\left(\frac{x_2}{y_2}\right) = 1.04 + (0.2)\left(\frac{0.4}{1.04}\right) = 1.11692$$

$$y_4 = y(x_4) = y(0.8) = y_3 + h\left(\frac{x_3}{y_3}\right) = 1.11692 + (0.2)\left(\frac{0.6}{1.11692}\right) = 1.22436.$$

When $h = 0.1$, we have

$$y_1 = y(x_1) = y(0.1) = y_0 + h\left(\frac{x_0}{y_0}\right) = 1 + (0.2)(0) = 1.0$$

$$y_2 = y(x_2) = y(0.2) = y_1 + h\left(\frac{x_1}{y_1}\right) = 1 + (0.1)\left(\frac{0.1}{1.0}\right) = 1.01.$$

$$y_3 = y(x_3) = y(0.3) = y_2 + h\left(\frac{x_2}{y_2}\right) = 1.01 + (0.1)\left(\frac{0.2}{1.01}\right) = 1.02980.$$

$$y_4 = y(x_4) = y(0.4) = y_3 + h\left(\frac{x_3}{y_3}\right) = 1.02980 + (0.1)\left(\frac{0.3}{1.02980}\right) = 1.05893.$$

$$y_5 = y(x_5) = y(0.5) = y_4 + h\left(\frac{x_4}{y_4}\right) = 1.05893 + (0.1)\left(\frac{0.4}{1.05893}\right) = 1.09670.$$

$$y_6 = y(x_6) = y(0.6) = y_5 + h\left(\frac{x_5}{y_5}\right) = 1.09670 + (0.1)\left(\frac{0.5}{1.09670}\right) = 1.14229.$$

$$y_7 = y(x_7) = y(0.7) = y_6 + h\left(\frac{x_6}{y_6}\right) = 1.14229 + (0.1)\left(\frac{0.6}{1.14229}\right) = 1.19482.$$

$$y_8 = y(x_8) = y(0.8) = y_7 + h\left(\frac{x_7}{y_7}\right) = 1.19482 + (0.1)\left(\frac{0.7}{1.19482}\right) = 1.25341.$$

The exact solution is $y = \sqrt{x^2+1}$. At $x = 0.8$, the exact value is $y(0.8) = \sqrt{1.64} = 1.28062$. The magnitudes of the errors in the solutions are the following:

$$h = 0.2 : |1.28062 - 1.22436| = 0.05626.$$

$$h = 0.1 : |1.28062 - 1.25341| = 0.02721.$$

Quiz

Question 1: You are given the differential equation $y' = 6x$ where $y = 2$ for $x = 0$. The statement: $y = 2$ for $x = 0$ is called ......
Question 2: Solve the initial value problem $yy' = x, \quad y(0) = 1$, using the Euler method in $0 \le x \le 0.8$, with $h = 0.2$ and $h = 0.1$. Compare the results with the exact solution at $x = 0.8$.

3.12 Modified Euler's Method

■ **Example 3.29** Solve the following initial value problem using the modified Euler method with $h = 0.1$ for $x \in [0, 0.3]$.

$$y' = y + x, \quad y(0) = 1.$$

Compare with the exact solution $y(x) = 2e^x - x - 1$. ■

Solution: Modified Euler's method is given by

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right)$$

We have $y' = f(x,y) = y + x, x_0 = 0, y_0 = 1$ and $h = 0.1$. Therefore,

$$y(0.1) = y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right) = 1.0 + 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} f(0,1)\right)$$

$$y_1 = 1.0 + 0.1 f\left(0 + 0.05, 1 + 0.05(1+0)\right) = 1 + 0.1 f\left(0.05, 1.05\right)$$

$$y_1 = 1 + 0.1(1.05 + 0.05) = 1.11.$$

Now, we have $x_1 = 0.1, y_1 = 1.11, y_1' = f(x_1, y_1) = y_1 + x_1 = 1.11 + 0.1 = 1.21.$

$$y(0.2) = y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right) =$$
$$1.11 + 0.1 f\left(0.1 + \frac{0.1}{2}, 1.11 + \frac{0.1}{2} f(0.1, 1.11)\right)$$

$$y_2 = 1.11 + 0.1 f\left(0.1 + 0.05, 1.11 + 0.05(1.11 + 0.1)\right) = 1.11 + 0.1 f\left(0.15, 1.1705\right)$$

$$y_1 = 1.11 + 0.1(1.1705 + 0.15) = 1.24205.$$

Again, we have $x_2 = 0.2, y_2 = 1.24205, y_2' = f(x_2) = y_2 + x_2 = 1.24205 + 0.2 = 1.44205.$

$$y(0.3) = y_3 = y_2 + hf\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2)\right) =$$
$$1.24205 + 0.1 f\left(0.2 + \frac{0.1}{2}, 1.24205 + \frac{0.1}{2} f(0.2, 1.24205)\right)$$

$$y_3 = 1.24205 + 0.1 f\left(0.2 + 0.05, 1.24205 + 0.05(1.44205)\right) = 1.24205 + 0.1 f\left(0.25, 1.31415\right)$$

$$y_1 = 1.11 + 0.1(1.31415 + 0.25) = 1.39846.$$

The exact solution at $x_1 = 0.1, y_1 = 1.11, h = 0.1$ is 1.11034, at $x_2 = 0.2, y_2 = 1.24205, h = 0.1$ is 1.24281 and $x_3 = 0.3, y_3 = 1.39846, h = 0.1$ is 1.39972. The magnitudes of the errors in the solutions are the following:

$$\text{At } x = 0.1 : |1.11034 - 1.11| = 0.00034.$$

$$\text{At } x = 0.2 : |1.24281 - 1.24205| = 0.00076.$$

$$\text{At } x = 0.3 : |1.39972 - 1.39846| = 0.00126.$$

■ **Example 3.30** For the following initial value problem, obtain approximations to $y(0.2)$ and $y(0.4)$, using the modified Euler method with $h = 0.2$.

$$y' = -2xy^2, \quad y(0) = 1.$$

Compare with the exact solution $y(x) = 1/(1+x^2)$. ■

Solution: Modified Euler's method is given by

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

We have $y' = f(x,y) = -2xy^2, x_0 = 0, y_0 = 1$ and $h = 0.2$. Therefore,

$$y(0.2) = y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0,y_0)\right) = 1.0 + 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}f(0,1)\right)$$

$$y_1 = 1.0 + 0.2f(0+0.1, 1+0.1(0)) = 1 + 0.2f(0.1,1)$$

$$y_1 = 1 + 0.2(-2(0.1)(1)^2 = 1 - 0.04 = 0.96.$$

Now, we have $x_1 = 0.2, y_1 = 0.96, y_1' = f(x_1,y_1) = -2.x_1.y_1^2 = -2(0.2)(0.96)^2 = \breve{}0.36864$

$$y(0.4) = y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1,y_1)\right) =$$
$$0.96 + 0.2f\left(0.2 + \frac{0.2}{2}, 0.96 + \frac{0.2}{2}f(0.2,0.96)\right)$$

$$y_2 = 0.96 + 0.2f(0.2+0.1, 0.96+0.1(\breve{}0.36864)) = 0.96 + 0.2f(0.3, 0.92314)$$

$$y_1 = 0.96 + 0.2(-2)(0.3)(0.92314)^2 = 0.85774.$$

The exact solution at $x_1 = 0.2, y_1 = 0.96, h = 0.2$ is 0.96154, at $x_2 = 0.4, y_2 = 0.85774, h = 0.2$ is 0.86207 The magnitudes of the errors in the solutions are the following:

$$\text{At } x = 0.2 : |0.96154 - 0.96| = 0.00154.$$

$$\text{At } x = 0.4 : |0.86207 - 0.85774| = 0.00433.$$

Solution of ordinary differential equations by Runge-Kutta method:

Second order Runge-Kutta method: Consider a first order initial value problem defined as

$$y' = f(x,y), y(x_0) = y_0$$

The second order Runge-Kutta method is defined as

$$y_1 = y_0 + \tfrac{1}{2}\{k_1 + k_2\}$$

where

$$k_1 = hf(x_0,y_0),$$
$$k_2 = hf(x_0+h, y_0+k_1)$$

Fourth order Runge-Kutta method: The fourth order Runge-Kutta method:

$$y_1 = y_0 + \tfrac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$k_1 = hf(x_0,y_0),$$
$$k_2 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2})$$
$$k_3 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2})$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

■ **Example 3.31** Solve the initial value problem

$$y' = -2xy^2, \quad y(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$. Use (i) second order Runge-Kutta method;
(ii) the fourth order classical Runge-Kutta method. Compare with the exact solution $y(x) = 1/(1+x^2)$. ■

Solution: We have, $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x,y) = -2xy^2$

Second order Runge-Kutta Method

$$y_1 = y_0 + \tfrac{1}{2}\{k_1 + k_2\}$$

where

$$k_1 = hf(x_0, y_0) = 0.2(-2x_0y_0^2) = (-0.4)(0)(1) = 0$$

and

$$k_2 = hf(x_0+h, y_0+k_1) = h(-2)(x_0+h)(y_0+k_1)^2 = 0.2(-2)(0+0.2)(1+0)^2 = -0.08$$

Therefore, by second order Runge-Kutta becomes

$$y_1 = y(0.2) = y_0 + \tfrac{1}{2}\{k_1 + k_2\} = 1 + \tfrac{1}{2}\{0 - 0.08\} = 0.96$$

Now, we have $x_1 = 0.2, y_1 = 0.96$, then

$$y_2 = y_1 + \tfrac{1}{2}\{k_1' + k_2'\}$$

where

$$k_1' = hf(x_1, y_1) = 0.2(-2)(x_1)(y_1^2) = (-0.4)(0.2)(0.96)^2 = -0.73728$$

and

$$k_2' = hf(x_1+h, y_1+k_1') = h(-2)(x_1+h)(y_1+k_1')^2 = 0.2(-2)(0.2+0.2)(0.96-0.73728)^2 = -0.00794$$

Therefore, by second order Runge-Kutta becomes

$$y_2 = y(0.4) = y_1 + \tfrac{1}{2}\{k_1' + k_2'\} = 0.96 + \tfrac{1}{2}\{-0.73728 - 0.00794\} = 0.58739$$

Fourth order Runge-Kutta method: The fourth order Runge-Kutta method:

$$y_1 = y_0 + \tfrac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$k_1 = hf(x_0, y_0) = -2(0.2)(0)(1)^2 = 0,$$

$$k_2 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}) = -2h(x_0 + \tfrac{h}{2})(y_0 + \tfrac{k_1}{2})^2 = -2(0.2)(0+0.2/2)(1+0/2)^2 = -0.04,$$

$$k_3 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}) = -2(0.2)(0.1)(0.98)^2 = -0.038416,$$

$$k_4 = hf(x_0+h, y_0+k_3) = -2(0.2)(0.2)(0.961584)^2 = -0.0739715,$$

The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} = 1 + \frac{1}{6}[0.0 - 0.08 - 0.076832 - 0.0739715] = 0.9615328.$$

Now, we have $x_1 = 0, y_1 = 0.9615328$.

$$k_1' = hf(x_1, y_1) = -2(0.2)(0.2)(0.9615328)^2 = -0.0739636,$$

$$k_2' = hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_1}{2}) = -2(0.2)(0.3)(0.924551)^2 = -0.1025753,$$

$$k_3' = hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_2}{2}) = -2(0.2)(0.3)(0.9102451)^2 = -0.0994255,$$

$$k_4' = hf(x_1 + h, y_1 + k_3) = -2(0.2)(0.4)(0.86210734)^2 = -0.1189166,$$

$$y_2 = y_1 + \frac{1}{6}\{k_1' + 2k_2' + 2k_3' + k_4'\} =$$
$$0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051506 - 0.1988510 - 0.1189166] = 0.8620525$$

Absolute errors in second order Runge-Kutta method.

$$\text{At } x = 0.2 : |0.9615385 - 0.96| = 0.0015385.$$

$$\text{At } x = 0.4 : |0.8620690 - 0.86030| = 0.0017690.$$

Absolute errors in fourth order Runge-Kutta method.

$$\text{At } x = 0.2 : |0.9615385 - 0.9615328| = 0.0000057.$$

$$\text{At } x = 0.4 : |0.8620690 - 0.8620525| = 0.0000165.$$

■ **Example 3.32** Given $y' = x^3 + y, y(0) = 2$, compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ using the Runge-Kutta method of fourth order. ■

Solution: Here we have $x_0 = 0, y_0 = 2, h = 0.2$ and $f(x,y) = x^3 + y$
Fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$k_1 = hf(x_0, y_0) = h(x_0^3 + y_0) = 0.2(0+2) = 0.4,$$

$$k_2 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}) = 0.2f(0.1, 2.2) = (0.2)(2.201) = 0.4402,$$

$$k_3 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}) = 0.2f(0.1, 2.2201) = (0.2)(2.2211) = 0.44422,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 2.44422) = (0.2)(2.45222) = 0.490444,$$

The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} = 2 + \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444] = 2.443214.$$

Now, we have $x_1 = 0.2, y_1 = 2.443214$.

$$k_1' = hf(x_1, y_1) = 0.2f(0.2, 2.443214) = (0.2)(2.451214) = 0.490243,$$

$$k_2' = hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_1}{2}) = 0.2f(0.3, 2.443214 + 0.245122) = (0.2)(2.715336) = 0.543067,$$

$$k_3' = hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_2}{2}) = 0.2f(0.3, 2.443214 + 0.271534) = (0.2)(2.741748) = 0.548350,$$

$$k_4' = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 2.443214 + 0.548350) = (0.2)(3.055564) = 0.611113,$$

$$y_2 = y(0.4) = y_1 + \tfrac{1}{6}\{k_1' + 2k_2' + 2k_3' + k_4'\} =$$
$$2.443214 + \frac{1}{6}[0.490243 + 2(0.543067) + 2(0.548350) + 0.611113] = 2.990579.$$

Now, we have $x_2 = 0.4, y_2 = 2.990579$.

$$k_1'' = hf(x_2, y_2) = 0.2f(0.4, 2.990579) = (0.2)(3.054579) = 0.610916,$$

$$k_2'' = hf(x_2 + \tfrac{h}{2}, y_2 + \tfrac{k_1}{2}) == 0.2f(0.5, 2.990579 + 0.305458) = (0.2)(3.421037) = 0.684207,$$

$$k_3'' = hf(x_2 + \tfrac{h}{2}, y_2 + \tfrac{k_2}{2}) = 0.2f(0.5, 2.990579 + 0.342104) = (0.2)(3.457683) = 0.691537,$$

$$k_4'' = hf(x_2 + h, y_2 + k_3) = 0.2f(0.6, 2.990579 + 0.691537) = (0.2)(3.898116) = 0.779623.$$

$$y_3 = y(0.6) = y_2 + \tfrac{1}{6}\{k_1'' + 2k_2'' + 2k_3'' + k_4''\} =$$
$$2.990579 + \frac{1}{6}[0.610916 + 2(0.684207) + 2(0.691537) + 0.779623] = 3.680917.$$

3.13 Milne's Predictor-Corrector Formula

Milne's Predictor-Corrector Formula: Let the first order initial value ordinary differential equation is $\dfrac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$. Then the Milne's Predictor Formula is defined as

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. In particular, this method requires the starting values y_0, y_1, y_2 and y_3. and the Milne's corrector Formula is defined as

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

Here $f_i = f(x_i, y_i)$, $f_{i-1} = f(x_{i-1}, y_{i-1}), \ldots$

■ **Example 3.33** Given $y' = x^3 + y, y(0) = 2$, the values $y(0.2) = 2.073, y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by Runge-Kutta method of fourth order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. ■

Solution: Milne's predictor-corrector method is given by

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}].$$

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.
We are given that

$$f(x,y) = x^3 + y, x_0 = 0, y_0 = 2, y(0.2) = y_1 = 2.073, y(0.4) = y_2 = 2.452, y(0.6) = y_3 = 3.023.$$

Predictor application
For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4h}{3}\left[2f_3 - f_2 + 2f_1\right].$$

We have

$$f_0 = f(x_0, y_0) = f(0,2) = 2, f_1 = f(x_1, y_1) = f(0.2, 2.073) = 2.081,$$

$$f_2 = f(x_2, y_2) = f(0.4, 2.452) = 2.516, f_3 = f(x_3, y_3) = f(0.6, 3.023) = 3.239.$$

$$y_4^{(0)} = 2 + \frac{4(0.2)}{3}\left[2(3.239) - 2.516 + 2(2.081)\right] = 4.1664.$$

Corrector application
First iteration For $i = 3$, we get

$$y_4^{(1)} = y_4^{(c)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(0)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(0)}) = f(0.8, 4.1664) = 4.6784.$

$$y_4^{(1)} = 2.452 + \frac{0.2}{3}\left[4.6784 + 4(3.239) + 2.516\right] = 3.79536.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(1)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(1)}) = f(0.8, 4.6784) = 4.30736.$

$$y_4^{(2)} = 2.452 + \frac{0.2}{3}\left[4.30736 + 4(3.239) + 2.516\right] = 3.770624.$$

We have $|y_4^{(2)} - y_4^{(1)}| = |3.770624 - 3.79536| = 0.024736.$
The result is accurate to one decimal place.
Third iteration

$$y_4^{(3)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(2)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(2)}) = f(0.8, 3.770624) = 4.282624.$

$$y_4^{(3)} = 2.452 + \frac{0.2}{3}\left[4.282624 + 4(3.239) + 2.516\right] = 3.768975.$$

We have $|y_4^{(3)} - y_4^{(2)}| = |3.768975 - 3.770624| = 0.001649$.
The result is accurate to two decimal place.
Fourth iteration

$$y_4^{(4)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(3)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(3)}) = f(0.8, 3.76897) = 4.280975$.

$$y_4^{(4)} = 2.452 + \frac{0.2}{3}[4.280975 + 4(3.239) + 2.516] = 3.768865.$$

We have $|y_4^{(4)} - y_4^{(3)}| = |3.768865 - 3.768975| = 0.000100$.
The result is accurate to three decimal place. The required result can be taken as $y(0.8) = 3.7689$.

■ **Example 3.34** Using Milne's predictor-corrector method, find $y(0.4)$ for the initial value problem $y' = x^2 + y^2, y(0) = 1$, with $h = 0.1$. Calculate all the required initial values by Euler's method. The result is to be accurate to three decimal places. ■

Solution: Milne's predictor-corrector method is given by

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}[2f_i - f_{i-1} + 2f_{i-2}].$$

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.
We are given that

$$f(x,y) = x^2 + y^2, x_0 = 0, y_0 = 1.$$

Euler's method gives

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.1(x_i^2 + y_i^2)$$

With $x_0 = 0, y_0 = 1$, we get

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 1.0 + 0.1(0 + 1.0) = 1.1.$$

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2) = 1.1 + 0.1(0.01 + 1.21) = 1.222.$$

$$y_3 = y_2 + 0.1(x_2^2 + y_2^2) = 1.222 + 0.1[0.04 + (1.222)^2] = 1.375328.$$

Predictor application
For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4h}{3}[2f_3 - f_2 + 2f_1].$$

We have

$$f_0 = f(x_0, y_0) = f(0,1) = 1, f_1 = f(x_1, y_1) = f(0.1, 1.1) = 1.22,$$

$$f_2 = f(x_2, y_2) = f(0.1, 1.222) = 1.533284, f_3 = f(x_3, y_3) = f(0.3, 1.375328) = 1.981527.$$

$$y_4^{(0)} = 1 + \frac{4(0.1)}{3}[2(1.981527) - 1.533284 + 2(1.22)] = 1.649303.$$

Corrector application
First iteration For $i = 3$, we get

$$y_4^{(1)} = y_4^{(c)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(0)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(0)}) = f(0.4, 1.649303) = 2.880200$.

$$y_4^{(1)} = 1.222 + \frac{0.1}{3}[2.880200 + 4(1.981527) + 1.533284] = 1.633320.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(1)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(1)}) = f(0.4, 1.633320) = 2.827734$.

$$y_4^{(2)} = 1.222 + \frac{0.1}{3}[2.827734 + 4(1.981527) + 1.533284] = 1.631571.$$

We have $|y_4^{(2)} - y_4^{(1)}| = |1.631571 - 1.633320| = 0.001749$.
The result is accurate to two decimal place.
Third iteration

$$y_4^{(3)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(2)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(2)}) = f(0.4, 1.631571) = 2.822024$.

$$y_4^{(3)} = 1.222 + \frac{0.1}{3}[2.822024 + 4(1.981527) + 1.533284] = 1.631381.$$

We have $|y_4^{(3)} - y_4^{(2)}| = |1.631381 - 1.631571| = 0.00019$.
The result is accurate to three decimal place. The required result can be taken as $y(0.4) = 1.63138$.

3.14 Adams-Bashforth Predictor-Corrector Formula

The Adams-Bashforth predictor-corrector method is given by
Predictor P: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.
Corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}\left[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2}.

■ **Example 3.35** Using the Adams-Bashforth predictor-corrector equations, evaluate $y(1.4)$, if y satisfies $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{x^2}$ and $y(1)=1, y(1.1)=0.996, y(1.2)=0.986, y(1.3)=0.972$. ■

Solution: Adams-Bashforth method of fourth order.

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}\left[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}.
Corrector C: Adams-Moulton method of fourth order.

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}\left[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2}.
We have $f(x,y)=\dfrac{1}{x^2}-\dfrac{y}{x}$, with $h=0.1$, we are given the values $y(1)=1, y(1.1)=0.996, y(1.2)=0.986, y(1.3)=0.972$.

Predictor application
For $i=3$, we obtain

$$y_4^{(p)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right].$$

We have

$$f_0 = f(x_0,y_0) = f(1,1) = 1-1 = 0, f_1 = f(x_1,y_1) = f(1.1, 0.996) = -0.079008,$$

$$f_2 = f(x_2,y_2) = f(1.2, 0.986) = -0.127222, f_3 = f(x_3,y_3) = f(1.3, 0.972) = -0.155976.$$

$$y_4^{(0)} = y_4^{(p)} = 0.972 + \frac{0.1}{24}\left[55(-0.155976) - 59(-0.127222) + 37(-0.079008) - 9(0)\right] = 0.955351.$$

Corrector application
First iteration For $i=3$, we get

$$y_4^{(c)} = y_4^{(1)} = y_3 + \frac{h}{24}\left[9f(x_4, y_4^{(0)}) + 19f_3 - 5f_2 + f_1\right].$$

Now, $f(x_4, y_4^{(0)}) = f(1.4, 0.955351) = -0.172189$.

$$y_4^{(1)} = 0.972 + \frac{0.1}{24}\left[9(-0.172189) + 19(-0.155976) - 5(-0.127222) + (-0.079008)\right] = 0.955516.$$

Second iteration

$$y_4^{(2)} = y_3 + \frac{h}{24}\left[9f(x_4, y_4^{(1)}) + 19f_3 - 5f_2 + f_1\right].$$

Now, $f(x_4, y_4^{(1)}) = f(1.4, 0.955516) = -0.172307$.

$$y_4^{(1)} = 0.972 + \frac{0.1}{24}\left[9(-0.172307) + 19(-0.155976) - 5(-0.127222) + (-0.079008)\right] = 0.955512.$$

Now, we have $|y_4^{(2)} - y_4^{(1)}| = |0.955512 - 0.955516| = 0.000004$.
Therefore, $y(1.4) = 0.955512$. The result is correct to five decimal places.

4. Power Series

4.1 What is a power series?

Many functions can be represented efficiently by means of infinite series. Examples we have seen in calculus include the exponential function

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \tag{4.1}$$

and the trigonometric functions,

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!}x^{2k}$$

and

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!}x^{2k+1}.$$

An infinite series of this type is called a power series. To be precise, a **power series** about x_0 is an infinite sum of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the a_n's are constants.