Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Milne's Predictor-Corrector Formula: Let the first order initial value ordinary differential equation is $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. Then the Milne's Predictor Formula is defined as

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}\left[2f_i - f_{i-1} + 2f_{i-2}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. In particular, this method requires the starting values y_0, y_1, y_2 and y_3. and the Milne's corrector Formula is defined as

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

Here $f_i = f(x_i, y_i)$, $f_{i-1} = f(x_{i-1}, y_{i-1}), \ldots$

Example

Given $y' = x^3 + y, y(0) = 2$, the values $y(0.2) = 2.073, y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by Runge-Kutta method of fourth order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$.

Solution: Milne's predictor-corrector method is given by

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}\left[2f_i - f_{i-1} + 2f_{i-2}\right].$$

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.
We are given that

$$f(x,y) = x^3 + y, x_0 = 0, y_0 = 2, y(0.2) = y_1 = 2.073, y(0.4) = y_2 = 2.452, y(0.6) = y_3 = 3.023.$$

Predictor application

For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4h}{3}\left[2f_3 - f_2 + 2f_1\right].$$

We have

$$f_0 = f(x_0, y_0) = f(0, 2) = 2, f_1 = f(x_1, y_1) = f(0.2, 2.073) = 2.081,$$

$$f_2 = f(x_2, y_2) = f(0.4, 2.452) = 2.516, f_3 = f(x_3, y_3) = f(0.6, 3.023) = 3.239.$$

$$y_4^{(0)} = 2 + \frac{4(0.2)}{3}\left[2(3.239) - 2.516 + 2(2.081)\right] = 4.1664.$$

Corrector application

First iteration For $i = 3$, we get

$$y_4^{(1)} = y_4^{(c)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(0)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(0)}) = f(0.8, 4.1664) = 4.6784.$

$$y_4^{(1)} = 2.452 + \frac{0.2}{3}[4.6784 + 4(3.239) + 2.516] = 3.79536.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(1)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(1)}) = f(0.8, 4.6784) = 4.30736.$

$$y_4^{(2)} = 2.452 + \frac{0.2}{3}\left[4.30736 + 4(3.239) + 2.516\right] = 3.770624.$$

We have $|y_4^{(2)} - y_4^{(1)}| = |3.770624 - 3.79536| = 0.024736.$
The result is accurate to one decimal place.

Third iteration

$$y_4^{(3)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(2)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(2)}) = f(0.8, 3.770624) = 4.282624.$

$$y_4^{(3)} = 2.452 + \frac{0.2}{3}[4.282624 + 4(3.239) + 2.516] = 3.768975.$$

We have $|y_4^{(3)} - y_4^{(2)}| = |3.768975 - 3.770624| = 0.001649.$
The result is accurate to two decimal place.

Fourth iteration

$$y_4^{(4)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(3)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(3)}) = f(0.8, 3.76897) = 4.280975.$

$$y_4^{(4)} = 2.452 + \frac{0.2}{3}\left[4.280975 + 4(3.239) + 2.516\right] = 3.768865.$$

We have $|y_4^{(4)} - y_4^{(3)}| = |3.768865 - 3.768975| = 0.000100.$ The result is accurate to three decimal place. The required result can be taken as $y(0.8) = 3.7689.$

Example

Using Milne's predictor-corrector method, find $y(0.4)$ for the initial value problem $y' = x^2 + y^2, y(0) = 1$, with $h = 0.1$. Calculate all the required initial values by Euler's method. The result is to be accurate to three decimal places.

Solution: Milne's predictor-corrector method is given by

$$y_{i+1}^{(p)} = y_{i-3} + \frac{4h}{3}\left[2f_i - f_{i-1} + 2f_{i-2}\right].$$

$$y_{i+1}^{(c)} = y_{i-1} + \frac{h}{3}\left[f(x_{i+1}, y_{i+1}^{(p)}) + 4f_i + f_{i-1}\right].$$

The method requires the starting values y_i, y_{i-1}, y_{i-2} and y_{i-3}. That is, we require the values y_0, y_1, y_2, y_3. Initial condition gives the value y_0.
We are given that

$$f(x,y) = x^2 + y^2, x_0 = 0, y_0 = 1.$$

Euler's method gives

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.1(x_i^2 + y_i^2)$$

With $x_0 = 0, y_0 = 1$, we get

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 1.0 + 0.1(0 + 1.0) = 1.1.$$

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2) = 1.1 + 0.1(0.01 + 1.21) = 1.222.$$

$$y_3 = y_2 + 0.1(x_2^2 + y_2^2) = 1.222 + 0.1[0.04 + (1.222)^2] = 1.375328.$$

Predictor application

For $i = 3$, we obtain

$$y_4^{(0)} = y_4^{(p)} = y_0 + \frac{4h}{3}\left[2f_3 - f_2 + 2f_1\right].$$

We have

$$f_0 = f(x_0, y_0) = f(0, 1) = 1, f_1 = f(x_1, y_1) = f(0.1, 1.1) = 1.22,$$

$$f_2 = f(x_2, y_2) = f(0.1, 1.222) = 1.533284, f_3 = f(x_3, y_3) = f(0.3, 1.375328) = 1.981527.$$

$$y_4^{(0)} = 1 + \frac{4(0.1)}{3}\left[2(1.981527)1.533284 + 2(1.22)\right] = 1.649303.$$

Corrector application

First iteration For $i = 3$, we get

$$y_4^{(1)} = y_4^{(c)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(0)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(0)}) = f(0.4, 1.649303) = 2.880200.$

$$y_4^{(1)} = 1.222 + \frac{0.1}{3}\left[2.880200 + 4(1.981527) + 1.533284\right] = 1.633320.$$

Second iteration

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(1)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(1)}) = f(0.4, 1.633320) = 2.827734.$

$$y_4^{(2)} = 1.222 + \frac{0.1}{3}[2.827734 + 4(1.981527) + 1.533284] = 1.631571.$$

We have $|y_4^{(2)} - y_4^{(1)}| = |1.631571 - 1.633320| = 0.001749.$
The result is accurate to two decimal place.

Third iteration

$$y_4^{(3)} = y_2 + \frac{h}{3}\left[f(x_4, y_4^{(2)}) + 4f_3 + f_2\right].$$

Now, $f(x_4, y_4^{(2)}) = f(0.4, 1.631571) = 2.822024.$

$$y_4^{(3)} = 1.222 + \frac{0.1}{3}[2.822024 + 4(1.981527) + 1.533284] = 1.631381.$$

We have $|y_4^{(3)} - y_4^{(2)}| = |1.631381 - 1.631571| = 0.00019.$ The result is accurate to three decimal place. The required result can be taken as $y(0.4) = 1.63138.$

Thanks !!!