Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Solution of ordinary differential equations by Runge-Kutta method:

Second order Runge-Kutta method: Consider a first order initial value problem defined as

$$y' = f(x, y), y(x_0) = y_0$$

The second order Runge-Kutta method is defined as

$$y_1 = y_0 + \frac{1}{2}\{k_1 + k_2\}$$

where

$$k_1 = hf(x_0, y_0),$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$

Fourth order Runge-Kutta method: The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$\begin{gathered} k_1 = hf(x_0, y_0), \\ k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) \\ k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) \\ k_4 = hf(x_0 + h, y_0 + k_3) \end{gathered}$$

Example

Solve the initial value problem

$$y' = -2xy^2, \quad y(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$. Use (i) second order Runge-Kutta method;
(ii) the fourth order classical Runge-Kutta method. Compare with the exact solution $y(x) = 1/(1 + x^2)$.

Solution: We have, $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = -2xy^2$

Second order Runge-Kutta Method

$$y_1 = y_0 + \frac{1}{2}\{k_1 + k_2\}$$

where

$$k_1 = hf(x_0, y_0) = 0.2(-2x_0y_0^2) = (-0.4)(0)(1) = 0$$

and

$$k_2 = hf(x_0 + h, y_0 + k_1) = h(-2)(x_0 + h)(y_0 + k_1)^2 = 0.2(-2)(0 + 0.2)(1 + 0)^2 = -0.08$$

Therefore, by second order Runge-Kutta becomes

$$y_1 = y(0.2) = y_0 + \frac{1}{2}\{k_1 + k_2\} = 1 + \frac{1}{2}\{0 - 0.08\} = 0.96$$

Now, we have $x_1 = 0.2, y_1 = 0.96$, then

$$y_2 = y_1 + \frac{1}{2}\{k_1' + k_2'\}$$

where

$$k_1' = hf(x_1, y_1) = 0.2(-2)(x_1)(y_1^2) = (-0.4)(0.2)(0.96)^2 = -0.73728$$

and

$$k_2' = hf(x_1 + h, y_1 + k_1') = h(-2)(x_1 + h)(y_1 + k_1')^2 = 0.2(-2)(0.2 + 0.2)(0.96 - 0.73728)^2 = -0.00794$$

Therefore, by second order Runge-Kutta becomes

$$y_2 = y(0.4) = y_1 + \frac{1}{2}\left\{k_1' + k_2'\right\} = 0.96 + \frac{1}{2}\left\{-0.73728 - 0.00794\right\} = 0.58739$$

Fourth order Runge-Kutta method: The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$k_1 = hf(x_0, y_0) = -2(0.2)(0)(1)^2 = 0,$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = -2h(x_0 + \frac{h}{2})(y_0 + \frac{k_1}{2})^2 = -2(0.2)(0 + 0.2/2)(1 + 0/2)^2 = -0.04,$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = -2(0.2)(0.1)(0.98)^2 = -0.038416,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = -2(0.2)(0.2)(0.961584)^2 = -0.0739715,$$

The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} = 1 + \frac{1}{6}[0.0 - 0.08 - 0.076832 - 0.0739715] = 0.9615328.$$

Now, we have $x_1 = 0, y_1 = 0.9615328.$

$$k_1' = hf(x_1, y_1) = -2(0.2)(0.2)(0.9615328)^2 = -0.0739636,$$

$$k_2' = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = -2(0.2)(0.3)(0.924551)^2 = -0.1025753,$$

$$k_3' = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = -2(0.2)(0.3)(0.9102451)^2 = -0.0994255,$$

$$k'_4 = hf(x_1 + h, y_1 + k_3) = -2(0.2)(0.4)(0.86210734)^2 = -0.1189166,$$

$$y_2 = y_1 + \frac{1}{6}\left\{k'_1 + 2k'_2 + 2k'_3 + k'_4\right\} = 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051506 - 0.1988510 - 0.1189166] = 0.8620525$$

Absolute errors in second order Runge-Kutta method.

$$\text{At } x = 0.2 : |0.9615385 - 0.96| = 0.0015385.$$

$$\text{At } x = 0.4 : |0.8620690 - 0.86030| = 0.0017690.$$

Absolute errors in fourth order Runge-Kutta method.

At $x = 0.2 : |0.9615385 - 0.9615328| = 0.0000057.$

At $x = 0.4 : |0.8620690 - 0.8620525| = 0.0000165.$

Example

Given $y' = x^3 + y, y(0) = 2$, compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ using the Runge-Kutta method of fourth order.

Solution: Here we have $x_0 = 0, y_0 = 2, h = 0.2$ and $f(x, y) = x^3 + y$

Fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

where

$$k_1 = hf(x_0, y_0) = h(x_0^3 + y_0) = 0.2(0 + 2) = 0.4,$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2f(0.1, 2.2) = (0.2)(2.201) = 0.4402,$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2f(0.1, 2.2201) = (0.2)(2.2211) = 0.44422,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 2.44422) = (0.2)(2.45222) = 0.490444,$$

The fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} = 2 + \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444] = 2.443214.$$

Now, we have $x_1 = 0.2, y_1 = 2.443214.$

$$k_1' = hf(x_1, y_1) = 0.2f(0.2, 2.443214) = (0.2)(2.451214) = 0.490243,$$

$$k_2' = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.2f(0.3, 2.443214 + 0.245122) = (0.2)(2.715336) = 0.543067,$$

$$k_3' = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.2f(0.3, 2.443214 + 0.271534) = (0.2)(2.741748) = 0.548350,$$

$$k_4' = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 2.443214 + 0.548350) = (0.2)(3.055564) = 0.611113,$$

$$y_2 = y(0.4) = y_1 + \frac{1}{6}\left\{k_1' + 2k_2' + 2k_3' + k_4'\right\} = 2.443214 + \frac{1}{6}[0.490243 + 2(0.543067) + 2(0.548350) + 0.611113] = 2.990579.$$

Now, we have $x_2 = 0.4, y_2 = 2.990579.$

$$k_1'' = hf(x_2, y_2) = 0.2f(0.4, 2.990579) = (0.2)(3.054579) = 0.610916,$$

$$k_2'' = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}) == 0.2f(0.5, 2.990579 + 0.305458) = (0.2)(3.421037) = 0.684207,$$

$$k_3'' = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}) = 0.2f(0.5, 2.990579 + 0.342104) = (0.2)(3.457683) = 0.691537,$$

$$k_4'' = hf(x_2+h, y_2+k_3) = 0.2f(0.6, 2.990579+0.691537) = (0.2)(3.898116) = 0.779623.$$

$$y_3 = y(0.6) = y_2 + \frac{1}{6}\left\{k_1''+2k_2''+2k_3''+k_4''\right\} = 2.990579 + \frac{1}{6}[0.610916+2(0.684207)+2(0.691537)+0.779623] = 3.680917.$$

Thanks !!!