Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Example

Solve the following initial value problem using the modified Euler method with $h = 0.1$ for $x \in [0, 0.3]$.

$$y' = y + x, \quad y(0) = 1.$$

Compare with the exact solution $y(x) = 2e^x - x - 1$.

Solution: Modified Euler's method is given by

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

We have $y' = f(x, y) = y + x, x_0 = 0, y_0 = 1$ and $h = 0.1$. Therefore,

$$y(0.1) = y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right) = 1.0 + 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}f(0, 1)\right)$$

$$y_1 = 1.0 + 0.1f\left(0 + 0.05, 1 + 0.05(1 + 0)\right) = 1 + 0.1f\left(0.05, 1.05\right)$$

$$y_1 = 1 + 0.1(1.05 + 0.05) = 1.11.$$

Now, we have $x_1 = 0.1, y_1 = 1.11, y_1' = f(x_1, y_1) = y_1 + x_1 = 1.11 + 0.1 = 1.21.$

$$y(0.2) = y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right) = 1.11 + 0.1f\left(0.1 + \frac{0.1}{2}, 1.11 + \frac{0.1}{2}f(0.1, 1.11)\right)$$

$$y_2 = 1.11 + 0.1f\left(0.1 + 0.05, 1.11 + 0.05(1.11 + 0.1)\right) = 1.11 + 0.1f\left(0.15, 1.1705\right)$$

$$y_1 = 1.11 + 0.1(1.1705 + 0.15) = 1.24205.$$

Again, we have $x_2 = 0.2, y_2 = 1.24205, y_2' = f(x_2) = y_2 + x_2 = 1.24205 + 0.2 = 1.44205.$

$$y(0.3) = y_3 = y_2 + hf\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right) = 1.24205 + 0.1f\left(0.2 + \frac{0.1}{2}, 1.24205 + \frac{0.1}{2}f(0.2, 1.24205)\right)$$

$$y_3 = 1.24205 + 0.1f\,(0.2 + 0.05, 1.24205 + 0.05(1.44205)) = 1.24205 + 0.1f\,(0.25, 1.31415)$$

$$y_1 = 1.11 + 0.1(1.31415 + 0.25) = 1.39846.$$

The exact solution at $x_1 = 0.1, y_1 = 1.11, h = 0.1$ is 1.11034, at $x_2 = 0.2, y_2 = 1.24205, h = 0.1$ is 1.24281 and $x_3 = 0.3, y_3 = 1.39846, h = 0.1$ is 1.39972. The magnitudes of the errors in the solutions are the following:

$$\text{At } x = 0.1 : |1.11034 - 1.11| = 0.00034.$$

$$\text{At } x = 0.2 : |1.24281 - 1.24205| = 0.00076.$$

$$\text{At } x = 0.3 : |1.39972 - 1.39846| = 0.00126.$$

Example

For the following initial value problem, obtain approximations to $y(0.2)$ and $y(0.4)$, using the modified Euler method with $h = 0.2$.

$$y' = -2xy^2, \quad y(0) = 1.$$

Compare with the exact solution $y(x) = 1/(1+x^2)$.

Solution: Modified Euler's method is given by

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

We have $y' = f(x, y) = -2xy^2, x_0 = 0, y_0 = 1$ and $h = 0.2$. Therefore,

$$y(0.2) = y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right) = 1.0 + 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} f(0, 1)\right)$$

$$y_1 = 1.0 + 0.2f\,(0 + 0.1, 1 + 0.1(0)) = 1 + 0.2f\,(0.1, 1)$$

$$y_1 = 1 + 0.2(-2(0.1)(1)^2 = 1 - 0.04 = 0.96.$$

Now, we have $x_1 = 0.2, y_1 = 0.96, y_1' = f(x_1, y_1) = -2.x_1.y_1^2 = -2(0.2)(0.96)^2 = 0.36864$

$$y(0.4) = y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right) = 0.96 + 0.2f\left(0.2 + \frac{0.2}{2}, 0.96 + \frac{0.2}{2}f(0.2, 0.96)\right)$$

$$y_2 = 0.96 + 0.2f\left(0.2 + 0.1, 0.96 + 0.1(0.36864)\right) = 0.96 + 0.2f\left(0.3, 0.92314\right)$$

$$y_1 = 0.96 + 0.2(-2)(0.3)(0.92314)^2 = 0.85774.$$

The exact solution at $x_1 = 0.2, y_1 = 0.96, h = 0.2$ is 0.96154, at $x_2 = 0.4, y_2 = 0.85774, h = 0.2$ is 0.86207 The magnitudes of the errors in the solutions are the following:

$$\text{At } x = 0.2 : |0.96154 - 0.96| = 0.00154.$$

$$\text{At } x = 0.4 : |0.86207 - 0.85774| = 0.00433.$$

Thanks !!!