Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Let the data

x	x_0	x_1	x_2	...	x_n
f(x)	$f(x_1)$	$f(x_2)$	$f(x_2)$	...	$f(x_n)$

be given at distinct unevenly spaced points or non-uniform points $x_0, x_1, ..., x_n$. This data may also be given at evenly spaced points. For this data, we can fit a unique polynomial of degree $\leq n$. Since the interpolating polynomial must use all the ordinates $f(x_0), f(x_1), ...f(x_n)$, it can be written as a linear combination of these ordinates. That is, we can write the polynomial as

$$P_n(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + ... + l_n(x)f(x_n)$$

where

$$l_i(x) = \frac{(x - x_0), (x - x_1), (x - x_2), ..., (x - x_{i-1}), (x - x_{i+1}), ..., (x - x_n)}{(x_i - x_0), (x_i - x_1), (x_i - x_2), ..., (x_i - x_{i-1}), (x_i - x_{i+1}), ..., (x_i - x_n)}$$

Example

Use Lagrange's formula, to find the quadratic polynomial that takes the values

x	0	1	3
f(x)	0	1	0

Solution: Since $f(x_0)$ and $f(x_2)$ are zero, we need to compute $l_1(x)$ only. We have

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x^2-3x)$$

The Lagrange quadratic polynomial is given by

$$\begin{aligned} P_2(x) = f(x) &= l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) = \\ 0 + -\frac{1}{2}(x^2-3x)(1) + 0 &= \frac{1}{2}(3x-x^2). \end{aligned}$$

Example

Construct the Lagrange interpolation polynomial for the data

x	-1	1	4	7
$f(x)$	-2	0	63	342

Hence, interpolate at $x = 5$.

Solution: Since $f(x_1)$ is zero, we need to compute $l_0(x)$, $l_2(x)$, $l_3(x)$ only. We have

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-4)(x-7)}{(-1-1)(-1-4)(-1-7)} = -\frac{1}{80}(x^3 - 12x^2 + 39x - 28).$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-1)(x-7)}{(4+1)(4-1)(4-7)} = -\frac{1}{45}(x^3-7x^2-x+7).$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)} = \frac{1}{144}(x^3-4x^2-x+4).$$

The Lagrange quadratic polynomial is given by

$$
\begin{aligned}
f(x) &= l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) + l_3 f(x_3) \\
&= -\frac{1}{80}(x^3 - 12x^2 + 39x - 28)(-2) + 0 - \frac{1}{45}(x^3 - 7x^2 - x + 7)(63) + \frac{1}{144}(x^3 - 4x^2 \\
&= \left(\frac{1}{40} - \frac{7}{5} + \frac{171}{72}\right) x^3 + \left(-\frac{3}{10} + \frac{49}{5} - \frac{171}{18}\right) x^2 + \left(\frac{39}{40} + \frac{7}{5} - \frac{171}{72}\right) x + \left(-\frac{7}{10} - \right. \\
&= x^3 - 1.
\end{aligned}
$$

Hence, $f(5) = P_3(5) = 5^3 - 1 = 124.$

Example

Given that $f(0) = 1, f(1) = 3, f(3) = 55$, find the unique polynomial of degree 2 or less, which fits the given data.

Solution: We have $x_0 = 0, f(x_0) = 1, x_1 = 1, f(x_1) = 3, x_2 = 3, f(x_2) = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2-4x+3).$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{1}{2}(3x-x^2).$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x^2-x).$$

The Lagrange quadratic polynomial is given by

$$\begin{aligned} P_2(x) = f(x) &= l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) \\ &= \frac{1}{3}(x^2 - 4x + 3)(1) + \frac{1}{2}(3x - x^2)(3) + \frac{1}{6}(x^2 - x)(55) \\ &= 8x^2 - 6x + 1. \end{aligned}$$

Quiz

Question 1: Using the data $sin(0.1) = 0.09983$ and $sin(0.2) = 0.19867$, find an approximate value of $sin(0.15)$ by Lagrange interpolation.

Question 2: Give two uses of interpolating polynomials.

Question 3: Write the property satisfied by Lagrange fundamental polynomials $l_i(x)$.

Thanks !!!