Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Definition

Central difference operator δ: The central difference operator is defined as

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$$

Definition

Average (Mean) operator μ: The central difference operator is defined as

$$\mu f(x) = \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2}$$

Prove that: $(i) \quad \Delta = E - 1 \quad (ii) \quad \Delta - \nabla = \nabla\Delta$

$(iii) \quad (1+\Delta)(1-\nabla) = 1$

$(iv) \quad \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \quad (v) \quad \delta = \nabla(1-\nabla)^{-1/2}$

$(vi) \quad \mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$

$(vii) \quad E = e^{hD}$, where $D = \frac{d}{dx}$.

Proof: (i) We know that $Ef(x) = f(x+h)$. Therefore,

$$\Delta f(x) = f(x+h) - f(x) \tag{1}$$

$$\implies \Delta f(x) = Ef(x) - f(x) \tag{2}$$

$$\implies \Delta = E - 1 \tag{3}$$

(ii) L.H.S

$$
\begin{aligned}
(\Delta - \nabla) f(x) &= \Delta f(x) - \nabla f(x) \\
&= \{f(x+h) - f(x)\} - \{f(x) - f(x-h)\} \\
&= f(x+h) - 2f(x) + f(x-h)
\end{aligned}
$$

R.H.S

$$
\begin{aligned}
(\nabla\Delta)f(x) &= \nabla\left\{\Delta f(x)\right\} \\
&= \nabla\left\{f(x+h)-f(x)\right\} \\
&= \left\{\nabla f(x+h)-\nabla f(x)\right\} \\
&= \left\{(f(x+h)-f(x))-(f(x)-f(x-h)\right\} \\
&= f(x+h)-2f(x)+f(x-h)
\end{aligned}
$$

Thus $L.H.S. = R.H.S.$

(iii) We know that $\nabla f(x) = f(x) - f(x-h) \implies \nabla f(x) = f(x) - E^{-1}f(x) \implies \nabla = 1 - E^{-1} \implies 1 - \nabla = E^{-1}$ and $\Delta = E - 1 \implies 1 + \Delta = E$. Therefore,

$$(1+\Delta)(1-\nabla) = (E)(E^{-1}) = 1$$

(iv) We know that

$$\begin{aligned} \mu f(x) \quad & \quad (4) \\ &= \left[\frac{1}{2} f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right] \\ &= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right] \\ \mu f(x) &= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x). \end{aligned}$$

(v) We know that $1-\nabla = E^{-1}$Therefore,

$$\begin{aligned} R.H.S. &= \nabla(1-\nabla)^{-1/2}f(x) \\ &= \nabla(E^{-1})^{-1/2}f(x) \\ &= \nabla\left\{(E^{1/2})f(x)\right\} = \nabla f(x+\frac{h}{2}) \\ &= f(x+\frac{h}{2}) - f(x+\frac{h}{2}-h) \\ &= f(x+\frac{h}{2}) - f(x+\frac{-h}{2}) \\ &= \delta f(x) = L.H.S. \end{aligned}$$

(vi) Try Yourself.

(v) We know that the Taylor's series

$$
\begin{aligned}
Ef(x) = f(x+h) & \\
&= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \ldots \\
&= f(x) + hDf(x) + \frac{h^2}{2}D^2f(x) + \ldots \\
&= e^{\left[1 + hD + \frac{h^2}{2}D^2 + \ldots\right]} f(x) \\
Ef(x) &= e^{hD}f(x).
\end{aligned}
$$

Thanks !!!