Mathematics-II (Numerical Methods)
Lecture Notes
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by

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Let the data $(x_i, f(x_i))$ be given with uniform spacing, that is, the nodal points are given by $x_i = x_0 + ih, i = 0, 1, 2, ..., n.$
Now we define several finite difference operators and relation between these finite difference operators.
Notation: We use the following notations as follows:

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_i = x_0 + ih, \text{ and}$$
$$f_0 = f(x_0), f_1 = f(x_1), f_2 = f(x_2), ..., f_i = f(x_i), ...$$

Definition

Shift Operator E: The Shift operator E is defined as

$$Ef(x) = f(x+h)$$

In particular,
$Ef(x_0) = f(x_0 + h) = f(x_1), Ef(x_1) = f(x_0 + 2h) = f(x_2), \ldots, Ef(x_i) = f(x_0 + (i+1)h) = f(x_{i+1}), \ldots$

Therefore, the operator E when applied on $f(x)$ shifts it to the value at the next nodal point. We have

$$E^2 f(x) = E\left(Ef(x)\right) = E\left(f(x+h)\right) = f(x+2h).$$

In general, we have

$$E^k f(x) = f(x + kh), \text{ where } k \text{ is any real number.}$$

For example: $E^{1/2}[f(x)] = f(x + \frac{1}{2}h).$

Definition

Forward Operator Δ: The forward operator Δ is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

In particular,

$$\begin{aligned}
\Delta f(x_0) &= f(x_0+h) - f(x_0) = f(x_1) - f(x_0), \\
\Delta f(x_1) &= f(x_0+2h) - f(x_0+h) = f(x_2) - f(x_1), \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\Delta f(x_i) &= f(x_0+(i+1)h) - f(x_0+ih) = f(x_{i+1}) - f(x_i),
\end{aligned}$$

These differences are called the first forward differences.

The second forward difference is defined by

$$
\begin{aligned}
\Delta^2 f(x) &= \Delta\left(\Delta f(x)\right) = \Delta\left(f(x+h) - f(x)\right) = \Delta f(x+h) - \\
&= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\} \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}
$$

The forward differences can be written in a tabular form as in Table 1

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x$
x_0	$f(x_0)$			
		$\Delta f(x_0) = f(x_1) - f(x_0)$		
x_1	$f(x_1)$		$\Delta^2 f(x_0) = \Delta f(x_1) - \Delta f(x_0)$	
		$\Delta f(x_1) = f(x_2) - f(x_1)$		$\Delta^3 f(x_0)$ $\Delta^2 f(x_1) - \Delta$
x_2	$f(x_2)$		$\Delta^2 f(x_1) = \Delta f(x_2) - \Delta f(x_1)$	

Example

Construct the forward difference table for the data

x	-1	0	1	2
$f(x)$	-8	3	1	12

Solution: We have the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-8			
		$3-(-8)=11$		
0	3		$-2-11=-13$	
		$1-3=-2$		$13+13=26$
1	1		$11+2=13$	
		$12-1=11$		
2	12			

Table: Forward Difference Table

Definition

Backward Difference Operator ∇: The Backward difference operator ∇ is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

In particular,

$$\begin{aligned}
\nabla f(x_1) &= f(x_0+h) - f(x_0) = f(x_1) - f(x_0), \\
\nabla f(x_2) &= f(x_0+2h) - f(x_0+h) = f(x_2) - f(x_1), \\
&\cdot \\
&\cdot \\
&\cdot \\
\nabla f(x_{i+1}) &= f(x_0+(i+1)h) - f(x_0+ih) = f(x_{i+1}) - f(x_i),
\end{aligned}$$

The second backward difference is defined by

$$
\begin{aligned}
\nabla^2 f(x) &= \nabla\left(\nabla f(x)\right) = \nabla\left(f(x) - f(x-h)\right) = \nabla f(x) - \nabla f(x \\
&= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\} \\
&= f(x) - 2f(x-h) + f(x-2h)
\end{aligned}
$$

The backward differences can be written in a tabular form as in Table 3

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x$
x_0	$f(x_0)$			
		$\nabla f(x_0) = f(x_1) - f(x_0)$		
x_1	$f(x_1)$		$\nabla^2 f(x_0) = \nabla f(x_1) - \nabla f(x_0)$	
		$\nabla f(x_1) = f(x_2) - f(x_1)$		$\nabla^3 f(x_0)$ $\nabla^2 f(x_1) - \nabla$
x_2	$f(x_2)$		$\nabla^2 f(x_1) = \nabla f(x_2) - \nabla f(x_1)$	

Example

Construct the backward difference table for the data

x	-1	0	1	2
$f(x)$	-8	3	1	12

Solution: We have the following difference table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
-1	-8			
		$3-(-8)=11$		
0	3		$-2-11=-13$	
		$1-3=-2$		$13+13=26$
1	1		$11+2=13$	
		$12-1=11$		
2	12			

Table: Backward Difference Table

Thanks !!!