Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

A polynomial equation of the form

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0 = 0$$

is called an **algebraic equation**.

For Example: $3x^5 + 2x^3 - x^2 + 35 = 0$, $x^4 + 5x^2 + 7 = 0$, $-2x^2 - 3^x + 4 = 0$,

Definition

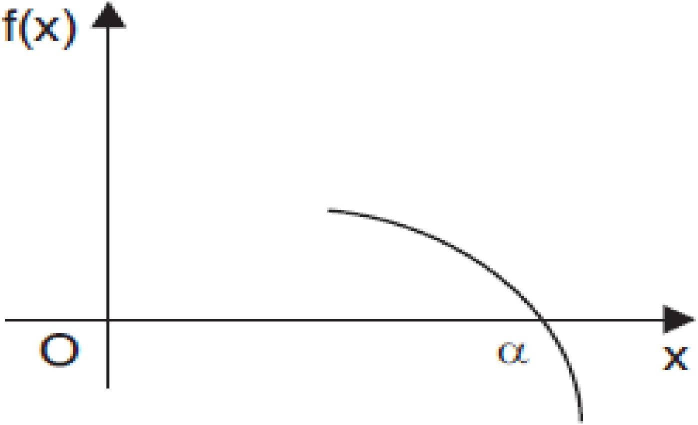
An equation which contains polynomials, exponential functions, logarithmic functions, trigonometric functions etc. is called a **transcendental equation**.

For Example: $xe^x - 2x = 0$, $x \tan x - \log x = 4$, $\sin^2 x + \cos x = 0$ are transcendental equations.

Definition

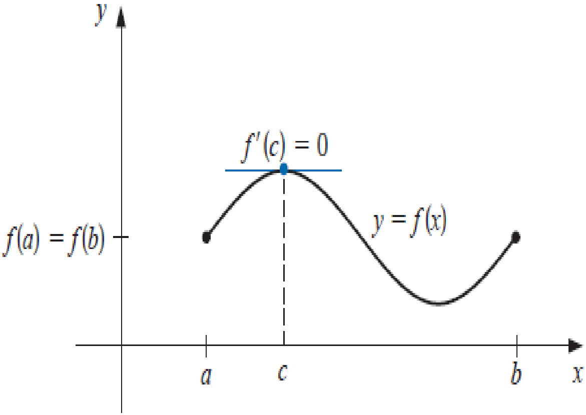
Root/zero: A number α, for which $f(\alpha) \equiv 0$ is called a root of the equation $f(x) = 0$, or a zero of $f(x)$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x-axis.

r



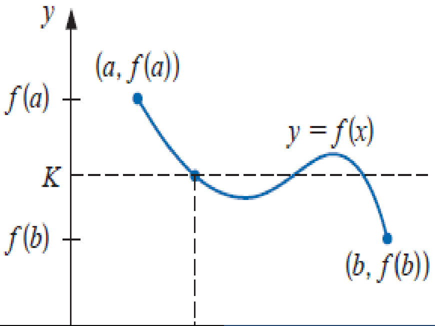
Theorem

Suppose the function f is continuous in $[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$.


y
$f'(c) = 0$
$f(a) = f(b)$
$y = f(x)$
a
c
b
x

Theorem

Intermediate Value Theorem: *If $f(x)$ is continuous on some interval $[a, b]$ and $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots in the interval (a, b).*



Example

Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $(0, 1)$.

Solution: Consider the function defined by $x^5 - 2x^3 + 3x^2 - 1 = 0$. The function f is continuous on $[0, 1]$. In addition, Here $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Therefore by, Intermediate Value Theorem there exist a number x with $0 < x < 1$, for which $x^5 - 2x^3 + 3x^2 - 1 = 0$. Hence the given function has the solution in the interval $(0, 1)$.

Bisection Method This method is applicable for numerically solving the equation $f(x) = 0$ for the real variable x, where f is a continuous function defined on an interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs. Then by the intermediate value theorem, the continuous function f must have at least one root in the interval (a, b). Now at each step, this method divides the interval in two interval by computing the midpoint $c = (a + b)/2$ of the interval and the value of the function f at that point c. Unless c is itself a root, there are now only two possibilities: either $f(a)$ and $f(c)$ have opposite signs or $f(c)$ and $f(b)$ have opposite signs. The method selects the subinterval that is guaranteed to be a root in the new interval. The process is continued until the interval is sufficiently small.

Explicitly, if $f(a)$ and $f(c)$ have opposite signs, then the method sets c as the new value for b, and if $f(b)$ and $f(c)$ have opposite signs then the method sets c as the new value for a. (If $f(c) = 0$ then c may be taken as the solution and the process stops.) In both cases, the new $f(a)$ and $f(b)$ have opposite signs, so the method is applicable to this smaller interval.

Example

Find the root of the equation $x^3 - x - 1 = 0$ by bisection method up to two places of decimal.

Solution: Here $f(x) = x^3 - x - 1$. Let $x_0 = 0$ so that $f(x_0 = 0) = -1 < 0$. and $x_1 = 2$ so that $f(x_1 = 2) = (2)^3 - (2) - 1 = 5 > 0$. Thus by intermediate value theorem the roots lies in the interval $(0, 2)$. By using bisection method, the first approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0 + 2}{2} = 1$$

Now $f(x_2 = 1) = (1)^3 - (1) - 1 = -1 < 0$. Since $f(x_1 = 2)f(x_2 = 1) = 5.(-1) = -5 < 0$. Therefore the roots lies in the interval $(x_2, x_1) i.e. (1, 2)$. Again by using bisection method, the second approximation is

Now $f\left(x_3 = \frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right) - 1 = \frac{7}{8} > 0$. Since $f(x_2 = 1)f(x_3 = 3/2) = (-1).(7/8) = -7/8 < 0$. Therefore the roots lies in the interval $(x_2, x_3) i.e. (1, 3/2)$. Again by using bisection method, the third approximation is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{1 + 3/2}{2} = \frac{5}{4}$$

Now $f\left(x_4 = \frac{5}{4}\right) = \left(\frac{5}{4}\right)^3 - \left(\frac{5}{4}\right) - 1 = -\frac{19}{64} < 0$. Since $f(x_4 = 5/4)f(x_3 = 3/2) = (-19/64).(7/8) = 133/512 > 0$. Therefore the roots lies in the interval $(x_4, x_3) i.e. (5/4, 3/2)$. Repeating this process we get
$x_5 = 11/8, x_6 = 21/16, x_7 = 43/32, x_8 = 85/64$. This process will be continue until the difference between last two approximation is less than 0.005.

Example

Using bisection method, find the root of the equation $3x - \sqrt{1 + \sin x} = 0$.

Solution: Here $f(x) = 3x - \sqrt{1 + \sin x}$. Let $x_0 = 0$ so that $f(x_0 = 0) = -1 < 0$. and $x_1 = 1$ so that $f(x_1 = 1) = 3(1) - \sqrt{1 + \sin(1)} = 3 - \sqrt{1 + 0.84} = 3 - 1.35 = 1.65 > 0$. Thus by intermediate value theorem the roots lies in the interval $(0, 1)$. By using bisection method, the first approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0 + 1}{2} = 1/2 = 0.5$$

Now $f(x_2 = 0.5) = 3(0.5) - \sqrt{1 + \sin(0.5)} = 3 - \sqrt{1 + 0.479} = 1.5 - 1.216 = 0.28 > 0$. Since $f(x_0 = 0) f(x_2 = 0.5) = (-1)(0.28) = -0.28 < 0$. Therefore the roots lies in the interval $(x_0, x_2) i.e. (0, 0.5)$. Again by using bisection method, the second approximation is

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0 + 0.5}{2} = 0.25$$

Now $f\,(x_3 = 0.25) = 3(0.25) - \sqrt{1 + \sin(0.25)} = 0.75 - \sqrt{1 + 0.247} = 0.75 - 1.216 = -0.117 < 0$. Since $f(x_2 = 0.5)f(x_3 = 0.25) = (0.28)(-0.117) = -0.33 < 0$. Therefore the roots lies in the interval $(x_2, x_3) i.e. (0.5, 0.25)$. Again by using bisection method, the third approximation is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.25}{2} = 0.35$$

Continuing this process we get the required answer.

Thanks !!!