Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

A zero of analytic function $f(x)$ is the value of z for which $f(z) = 0$.

Definition

SINGULAR POINT: A point at which a function $f(z)$ is not analytic is known as a singular point or singularity of the function.

For Example: The function $\dfrac{1}{z-a}$ has a singular point at $z - a = 0$ or $z = a$.

Definition

Isolated singular point: If $z = a$ is a singularity of $f(z)$ and if there is no other singularity in the neighborhood of the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$; otherwise it is called **non-isolated**.

For Example: The function $\dfrac{1}{(z-a)(z-b)}$ has a singular point at $z = a, b$. Here in the neighborhood of a and b, there does not exits any other singularities. Hence a and b are isolated singularities.

Example of non-isolated singularity: The function $f(z) = cosec\left(\frac{\pi}{z}\right)$ is not analytic at the points where $\sin\left(\frac{\pi}{z}\right) = 0$ i.e., at the points $\frac{\pi}{z} = n\pi$ i.e., the points $z = \frac{1}{n}$ i.e., the points $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Here $z = 0$ is the limit points of $z = \frac{1}{n}$. Hence $z = 0$ is the non-isolated singularity of the function $f(z) = cosec\left(\frac{\pi}{z}\right)$ because in the neighbourhood of z $= 0$, there are infinite number of other singularities $z = \frac{1}{n}$, when n is very large.

Definition

Laurent's series: An expansion of the function $f(z)$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

is called Laurent's series expansion. The part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ is called **Principal Part** of the function $f(z)$ at $z = 0$.

Definition

Pole: If the principle part of the function $f(z)$ at $z = a$ in Laurent's expansion has only finite number of terms (say m), we say $f(z)$ has pole of order m at $z = a$. or
if $\exists$ a $+$ve integer m such that

$$\lim_{z \to a} (z-a)^m f(z) = k(constant) \neq 0.,$$

then we say that $f(z)$ has a pole of order m at $z = a$.

For Example:1. The function $f(z) = \dfrac{1}{(z-1)^2(z+2)^5}$ has a pole at $z = 1$ of order 2 and has a pole at $z = -2$ of order 5.

2. $\tan z$ and $\sec z$ has simple poles at $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots.$

3. $\cot z$ and $cosecz$ has simple poles at $z = 0, \pm\pi, \pm 2\pi, \ldots.$

Definition

Essential Singularities: If the principle part of $f(z)$ at $z=a$ in Laurent's series expansion has infinite number of terms, then we say that $z=a$ is an essential singularities of $f(z)$. or
If $\lim_{z\to a} f(z)$ does not exist, then we say that $z=a$ is essential singularities $f(z)$.

For Example:1. The function $f(z) = e^{1/z}$ has an essential singularities at $z = 0$ because its expansion about $z = 0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

has infinite number of terms in negative powers of z.

2. The function $f(z) = \sin\left(\frac{1}{z-a}\right)$ has an essential singularities at $z = a$ because its expansion about $z = a$

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \ldots$$

has infinite number of terms in negative powers of $z - a$.

Definition

Removable Singularities: If the principle part of $f(z)$ at $z = a$ in Laurent's series expansion has no terms, then we say that $z = a$ is a removable singularities of $f(z)$. or $z = a$ is said to be removable singularities if $\lim_{z \to a} f(z)$ exist finitely.

For Example:1. The function $f(z) = \dfrac{\sin z}{z}$ has removal singularities at $z = 0$ because its expansion about $z = 0$

$$\frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 \ldots$$

has no number of terms in negative powers of z.

2. The function $f(z) = \dfrac{z - \sin z}{z^2}$ has a removable singularities at $z = 0$ because its expansion about $z = 0$

$$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right)\right] = \frac{z}{3!} - \frac{z^3}{5!} + \ldots$$

has no number of terms in negative powers of z.

Example

Find out the zeros and discuss the nature of the singularities of

$$f(z) = \frac{z-2}{z^2}\left(\sin\frac{1}{z-1}\right).$$

Solution: Poles of $f(z)$ are given by equating to zero the denominator of $f(z)$ i.e. $z = 0$ is a pole of order two.
zeros of $f(z)$ are given by equating to zero the numerator of $f(z)$ i.e., $(z-2)\sin\left(\frac{1}{z-1}\right) = 0$

$\implies$ Either $z - 2 = 0$ or $\sin\left(\frac{1}{z-1}\right) = 0$

$\implies$ $z = 2$ and $\frac{1}{z-1} = n\pi$

$\implies z = 2$ and $z = 1 + \dfrac{1}{n\pi}, n = \pm 1, \pm 2, \pm 3, \ldots$
Thus, $z = 2$ is a simple zero. The limit point of the zeros $z = 1 + \dfrac{1}{n\pi}$ are given by $z = 1$. Hence $z = 1$ is an isolated essential singularity.

Thanks !!!