Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

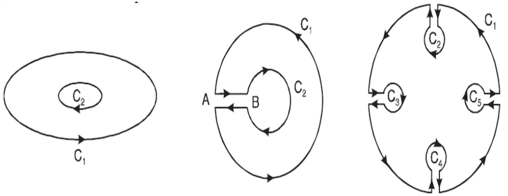
Simply connected Region: A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region D are the points of the region D.

Definition

Multi-Connected Region: Multi-connected region is bounded by more than one curve. We can convert a multi-connected region into a simply connected one, by giving it one or more cuts.

Definition

A function $f(z)$ is said to be meromorphic in a region R if it is analytic in the region R except at a finite number of poles.



Multi-Connected Region Simply Connected Region Simply Connected Region

Definition

Single-valued and Multi-valued function: If a function has only one value for a given value of z, then it is a single valued function.

$$\text{For example } f(z) = z^2$$

If a function has more than one value, it is known as multi-valued function.

Definition

Jordan arc: A continuous arc without multiple points is called a Jordan arc.

Definition

Regular arc: If the derivatives of the given function are also continuous in the given range, then the arc is called a regular arc.

Definition

Contour: A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.
The contour is said to be closed if the starting point A of the arc coincides with the end point B of the last arc.

Definition

Zeros of an Analytic function: The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$.

For example,

(1) Zeros of $z^2 - 3z + 2$ are $z = 1$ and $z = 2$.

(2) Zeros of $\cos z$ is $\pm(2n-1)\dfrac{\pi}{2}$, where $n = 1, 2, 3, \ldots$

Theorem

CAUCHY'S INTEGRAL THEOREM-I *If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C, then*

$$\int_C f(z)dz = 0$$

.

Proof: See the proof at page no. 548 in the book written by H.K.Dass

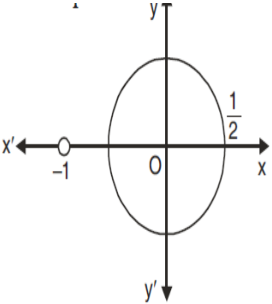
Note: If there is no pole inside and on the contour then the value of the integral of the function is zero.

Example

Find the integral $\int_C \frac{3z^2+7z+1}{z+1}dz$ where C is the circle $|z| = \frac{1}{2}$

Solution: Poles of the integrand are given by putting the denominator equal to zero.i.e.
$z+1=0 \implies z=-1$ The given circle $|z|=\frac{1}{2}$ with centre at $z=0$ and radius $\frac{1}{2}$ does not enclose any singularity of the given function. Therefore by Cauchy Integral Formula

$$\int_C \frac{3z^2+7z+1}{z+1}dz = 0.$$


y
1/2
x′
−1
O
x
y′

•

Theorem

CAUCHY'S INTEGRAL THEOREM-II *If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then, then*

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz = f(a).$$

, where C is any closed curve in R surrounding the point $z = a$.

Proof: See the proof at page no. 551 in the book written by H.K.Dass

Example

Evaluate the integral $\int_C \frac{1}{z^2+9}dz$ where C is the circle $|z+3i| = 2$ and $|z| = 5$.

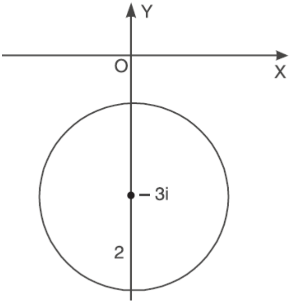
Solution: Here $f(z) = \dfrac{1}{z^2+9}$.

The poles of f (z) can be determined by equating the denominator equal to zero.

(i.) $z^2+9=0 \implies z = \pm 3i$. Pole at $z=-3i$ lies in the given circle C. $\int_C f(z)dz = \int_C \dfrac{1}{z^2+9} = \int_C \dfrac{1}{(z+3i)(z-3i)}$.

$$= \int_C \frac{1/(z-3i)}{(z+3i)} = 2\pi i \left[\frac{1}{(z-3i)}\right]_{z=-3i}$$

$$=2\pi i \left[\frac{1}{(-3i-3i)}\right] = -\frac{\pi}{3}.$$


Y
O
X
– 3i
2

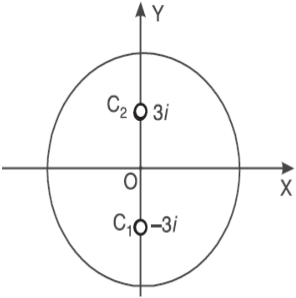
(ii.) $z^2+9=0 \implies z=\pm 3i$. Pole at $z=-3i$ lies in the given circle C. $\int_C f(z)dz = \int_C \frac{1}{z^2+9} = \int_C \frac{1}{(z+3i)(z-3i)}$.

$$= \int_C \frac{1/(z-3i)}{(z+3i)} + \int_C \frac{1/(z+3i)}{(z-3i)}$$

$$=2\pi i \left[\frac{1}{(z-3i)}\right]_{z=-3i} + 2\pi i \left[\frac{1}{(z+3i)}\right]_{z=3i}$$

$$=2\pi i \left[\frac{1}{(-3i-3i)}\right] + 2\pi i \left[\frac{1}{(3i+3i)}\right]$$

$$=- \frac{\pi}{3} + \frac{\pi}{3} = 0.$$


Y
C2
3i
O
X
C1
−3i

Theorem

CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION *If a function $f(z)$ is analytic in a region R, then its derivative at any point $z = a$ of R is also analytic in R, and is given by,*

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz.$$

Proof: See the proof at page no. 550 in the book written by H.K.Dass

Theorem

CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF ORDER n OF AN ANALYTIC FUNCTION *If a function $f(z)$ is analytic in a region R, then its derivative of order n at any point $z = a$ of R is also analytic in R, and is given by,*

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

Example

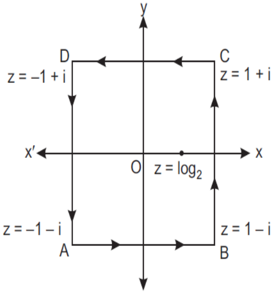
Find the integral $\int_C \frac{e^{3z}}{(z-\log 2)^4}dz$, where C is the square with vertices at $\pm 1, \pm i$.

Solution: Here $\int_C \frac{e^{3z}}{(z-\log 2)^4}dz$ Poles of the integrand are given by putting the denominator equal to zero.i.e. $(z-\log 2)^4 = 0 \implies z = \log 2.$ The integral has a pole of fourth order.

$$\int_C \frac{e^{3z}}{(z-\log 2)^4}dz = \frac{2\pi i}{3!} f''' \left[e^{3z}\right]_{z=\log 2}$$

$$= \frac{2\pi i}{3!} 3.3.3. \left[e^{3z}\right]_{z=\log 2}$$

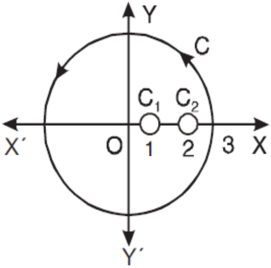
$$=9\pi i e^{3\log 2} = 9\pi i e^{\log 2^3} = 9\pi i e^{\log 8} = 72\pi i.$$


y
D
C
z = –1 + i
z = 1 + i
x'
x
O
z = log2
z = –1 – i
z = 1 – i
A
B

Example

Use Cauchy integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

Solution: Here $\int_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ Poles of the integrand are given by putting the denominator equal to zero.i.e. $(z-1)(z-2) = 0 \implies z = 1, 2.$ The integral has two pole at $z = 1, 2.$ The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the poles $z = 1$, and $z = 2$.


Y
C
C_1
C_2
X′
O
1
2
3
X
Y′

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$
$$= \int_{C_1} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-2)}{(z-1)} dz +$$
$$\int_{C_2} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-1)}{(z-2)} dz$$
$$= 2\pi i \left[\frac{(\sin \pi z^2 + \cos \pi z^2)}{(z-2)} \right]_{z=1} + 2\pi i \left[\frac{(\sin \pi z^2 + \cos \pi z^2)}{(z-1)} \right]_{z=2}$$
$$= 2\pi i \left[\frac{(\sin \pi + \cos \pi)}{(1-2)} \right] + 2\pi i \left[\frac{(\sin 4\pi + \cos 4\pi)}{(2-1)} \right] =$$
$$2\pi i \left[\frac{-1}{-1} \right] + 2\pi i \left[\frac{1}{1} \right] = 4\pi i.$$

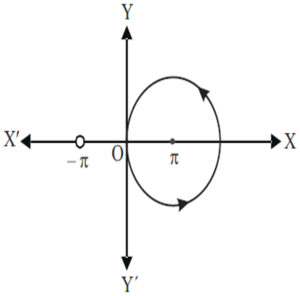
Which is the required value of the given integral.

Example

Use Cauchy integral formula to evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz$, where C is the circle $|z-\pi| = 3.2$.

Solution: Here $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz$, where C is a circle $|z-\pi| = 3.2$ with centre π and radius 3.2. Poles of the integrand are given by putting the denominator equal to zero.i.e.
$(z+\pi)^3 = 0 \implies z = -\pi, -\pi, -\pi$. The integral has a pole of order 3 at $z = \pi$. But there is no pole within C. By Cauchy Integral Formula $\int_C \frac{e^{3iz}}{(z+\pi)^3}dz = 0$.

Which is the required value of the given integral.


Y
X′
X
O
−π
π
Y′

Thanks !!!