Mathematics-II (Complex Variable)
Lecture Notes
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by

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Line Integral: If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + idy$,
we have

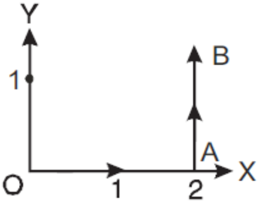
$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy),$$ where C is closed path,

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Example

Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis from $z = 0$ to $z = 2$ and then along a line parallel to y-axis from $z = 2$ to $z = 2 + i$.

Solution: $\int_0^{2+i}(\bar{z})^2dz = \int_0^{2+i}(x-iy)^2(dx+idy)$
$= \int_{OA}(x)^2dx + \int_{AB}(2-iy)^2idy$ Since [Along $OA, y=0, dy=0$, x varies 0 to 2. Along $AB, x=2, dx=0$ and y varies 0 to 1]
$= \int_0^2(x)^2dx + \int_0^1(2-iy)^2idy$
$= \int_0^2 x^2dx + i\int_0^1(4-4iy-y^2)dy$


Y
1
B
A
X
O
1
2

$$= \left[\frac{x^3}{3}\right]_0^2 + i\left[\left(4y - 4i\frac{y^2}{2} - \frac{y^3}{3}\right)\right]_0^1 = \frac{8}{3} + i\left(4 - 4i\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\left(14 + 11i\right).$$

Which is the required value of the given integral.

Example

Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path
$(a) \qquad y = x \qquad (b) \qquad y = x^2.$

Solution: (a) Along the line $y = x$,

$dy = dx$ so that $dz = dx + idy$

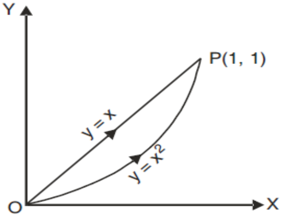
$$dz = dx + idx = (1+i)dx$$

By putting $y = x$ and $dz = (1+i)dx]$, we have

$$\int_0^{1+i}(x^2 - iy)dz = \int_0^{1+i}(x^2 - ix)dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1+i)\left[\frac{1^3}{3} - i\frac{1^2}{2}\right] = \frac{1}{6}(5-i).$$

Which is the required value of the given integral.


Y
P(1, 1)
y=x
y=x²
O
X

(b) Along the parabola $y = x^2$, $dy = 2xdx$ so that
$dz = dx + idy$
$\implies dz = dx + 2ixdx = (1 + 2ix)dx$ and x varies from 0 to 1.
$\int_0^{1+i}(x^2 - iy)dz = \int_0^1 (x^2 - ix^2)(1 + 2ix)dx$
$= (1-i)\int_0^1 (x^2 + 2ix^3)dx = (1-i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1 =$
$(1-i)\left[\frac{1^3}{3} + 2i\frac{1^4}{4}\right] = \frac{1}{6}(5+i).$
Which is the required value of the given integral.

Example

Evaluate $\int_C (z-a)^n dz$ where C is the circle with centre a and r. Discuss the case when $n=-1$.

Solution: The equation of circle C is $|z-a| = r$ or

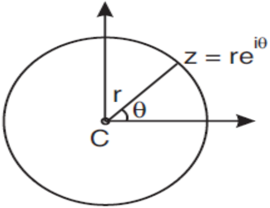
$$z - a = re^{i\theta}$$

where θ varies from 0 to 2π. so that $dz = rie^{i\theta}$

By putting $z - a = re^{i\theta}$ and $dz = rie^{i\theta}$, we have

$$\int_C (z-a)^n dz = \int_0^{2\pi} (re^{i\theta})^n rie^{i\theta} d\theta$$

$$= \int_0^{2\pi} r^{n+1} i e^{i(n\theta+\theta)} d\theta = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$


z = re^{iθ}
r
θ
C

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi-1}\right] =$$

$$\frac{r^{n+1}}{n+1}\left[\cos(n+1)2\pi + i\sin(n+1)2\pi - 1\right]$$

$$= \frac{r^{n+1}}{n+1}\left[1 + i.0 - 1\right] = 0.$$

When $n = -1$,

$$\int_C (z-a)^n dz = \int_C \frac{1}{(z-a)} dz = \int_0^{2\pi} \frac{1}{re^{i\theta}} rie^{i\theta} d\theta = \int_0^{2\pi} i d\theta =$$

$$2\pi i$$

Which is the required value of the given integral.

Thanks !!!