Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

BILINEAR TRANSFORMATION (Mobius Transformation) The transformation of the form

$$w = \frac{az+b}{cz+d}, \qquad \text{provided } ad - bc \neq 0.$$

is called bilinear transformation.

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INVARIANT POINTS OF BILINEAR TRANSFORMATION We know that

$$w = \frac{az+b}{cz+d},$$

If z maps into itself, then $w = z$

$$z = \frac{az+b}{cz+d}, \quad (1)$$

Roots of (1) are the invariants or fixed points of the bilinear transformation.
If the roots are equal, the bilinear transformation is said to be parabolic.

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CROSS-RATIO If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)},$$

is called the cross-ratio of z_1, z_2, z_3, z_4.

Theorem

A bilinear transformation preserves cross-ratio of four points i.e.

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

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Example

Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z| < 1$.

Solution: Let the required transformation be $w = \dfrac{az+b}{cz+d}$

$$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z + 1} \implies w = \frac{pz+q}{rz+1} \quad (2)$$

where $p = \dfrac{a}{d}$, $q = \dfrac{b}{d}$ and $r = \dfrac{c}{d}$.

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On substituting the values of $z = 1$ and corresponding values of $w = i$ in (2), we get

$$i = \frac{p+q}{r+1} \implies p + q = ir + i \quad (3)$$

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Again on substituting the values of $z = i$ and corresponding values of $w = 0$ in (2), we get

$$0 = \frac{ip + q}{ir + 1} \implies ip + q = 0 \tag{4}$$

Finally, on substituting the values of $z = -1$ and corresponding values of $w = -i$ in (2), we get

$$-i = \frac{-p + q}{-r + 1} \implies -p + q = ir - i \tag{5}$$

Solving equation (3), (4) and (5, we get $p = i$, $q = 1$ and $r = -i$.

Now substitute the value of p, q and r in (2), we get the required Bilinear transformation as

$$w = \frac{iz + 1}{-iz + 1}. \tag{6}$$

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Bilinear Transformation...

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Introduction

BILINEAR TRANSFORMATION (Mobius Transformation)

Again on substituting the values of $z = i$ and corresponding values of $w = 0$ in (2), we get

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To find the image of $|z| < 1$ under the Bilinear map $w = \dfrac{iz+1}{-iz+1}$, we rewrite the given equation in the terms of real and imaginary parts as

$$u + iv = \frac{i(x+iy)+1}{-i(x+iy)+1} = \frac{ix - y + 1}{-ix + y + 1} = \frac{(ix - y + 1)(ix + y -}{(-ix + y + 1)(ix + y}$$

Equating real parts we get

$$u = \frac{-x^2 - y^2 + 1}{x^2 + (y+1)^2}. \qquad (8)$$

But we have, $|z| < 1 \implies x^2 + y^2 < 1 \implies 0 < 1 - x^2 - y^2$. Thus equation (8) shows that $u > 0$. In other words the open disk in z-plane maps into open upper half of w-plane.

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