Mathematics-II (Complex Variable)
Lecture Notes
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by

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Transformation: For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this **transformation or mapping of z-plane into w-plane**. If a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

Example

Transform the rectangular region $ABCD$ in z-plane bounded by $x = 1, x = 3$; $y = 0$ and $y = 3$. Under the transformation $w = z + (2 + i)$.

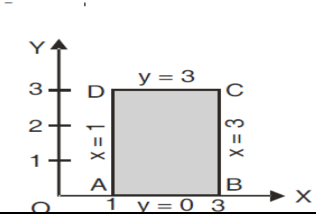
Solution: Here

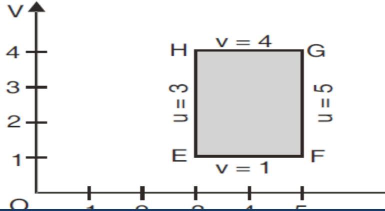
$$\begin{aligned} w &= z + (2 + i) \\ \implies u + iv &= x + iy + (2 + i) \\ &= (x + 2) + i(y + 1) \end{aligned}$$

By equating real and imaginary quantities, we have $u = x + 2$ and $v = y + 1$.

z-plane	w-plane	z-plane	w-plane
x	u = x + 2	y	v = y + 1
1	= 1 + 2 = 3	0	= 0 + 1 = 1
3	= 3 + 2 = 5	3	= 3 + 1 = 4

Here the lines $x = 1, x = 3$; $y = 0$ and $y = 1$ in the z-plane are transformed onto the line $u = 3, u = 5; v = 1$ and $v = 4$ in the w-plane. The region $ABCD$ in z-plane is transformed into the region $EFGH$ in w-plane.





Example

Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$.

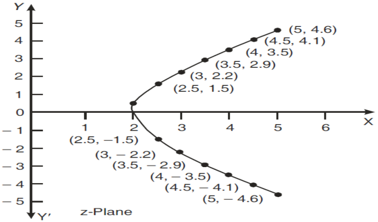
Solution.

$$\begin{aligned} w &= z^2 \\ \implies u + iv &= (x+iy)^2 \\ &= x^2 - y^2 + 2ixy \end{aligned}$$

This gives $u = x^2 - y^2$ and $v = 2xy$.

Table of (x, y) and (u, v)

x	2	2.5	3	3.5	4	4.5	5
$y = \pm\sqrt{x^2 - 4}$	0	± 1.5	± 2.2	± 2.9	± 3.5	± 4.1	± 4.6
$u = x^2 - y^2$	4	4	4	4	4	4	4
$v = 2xy$	0	± 7.5	± 13.2	± 20.3	± 28	± 36.9	± 46



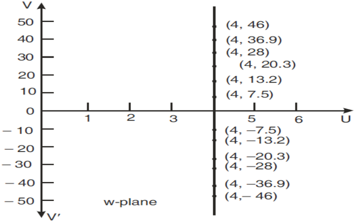


Image of the curve $x^2 - y^2 = 4$ is a straight line, $u = 4$ parallel

Let two curves C_1, C_2 in the z-plane intersect at the point Z_0 and the corresponding curve C_1^*, C_2^* in the w-plane intersect at $f(z_0)$. **If the angle of intersection of the curves at z_0 in z-plane is the same as the angle of intersection of the curves of w-plane at $f(z_0)$ in magnitude and sense, then the transformation is called conformal**.
If only the magnitude of the angle is preserved, transformation is **Isogonal**.

Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

Theorem

If $f(z)$ is analytic, mapping is conformal.

Theorem

Prove that an analytic function $f(z)$ ceases to be conformal at the points where $f'(z) = 0$.

Note 1. The point at which $f'(z) = 0$ is called a **critical point** of the transformation.

Example

If $u = 2x^2 + y^2$ and $v = \dfrac{y^2}{x}$, show that the curves $u = constant$ and $v = constant$ cut orthogonally at all intersections but that the transformation $w = u + iv$ is not conformal.

Solution: For the curve, $2x^2 + y^2 = u$

$$2x^2 + y^2 = constant = c_1(say) \tag{1}$$

Differentiating (1), we get

$$4x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2x}{y} = m_1(say) \tag{2}$$

For the curve, $\dfrac{y^2}{x} = \text{constant} = c_2$ (say),

$$y^2 = c_2 x \tag{3}$$

Differentiating (3), we get

$$2y\frac{dy}{dx} = c_2 \implies \frac{dy}{dx} = \frac{c_2}{2y} = \frac{y^2}{x} \times \frac{1}{2y} = \frac{y}{2x} = m_2(say) \tag{4}$$

For orthogonal, from equation (2) and (4), we have

$$m_1 m_2 = \left(\frac{-2x}{y}\right)\left(\frac{y}{2x}\right) = -1$$

Hence, two curves cut orthogonally.

However, since

$$\frac{du}{dx} = 4x,\ \frac{du}{dy} = 2y,\ \frac{dv}{dx} = -\frac{y^2}{x^2} \text{ and } \frac{dv}{dy} = \frac{2y}{x}$$

The Cauchy-Riemann equations are not satisfied by u and v. Hence, the function $u + iv$ is not analytic. So, the transformation is not conformal.

Example

For the conformal transformation $w = z^2$, show that

a. The coefficient of magnification at $z = 2 + i$ is $2\sqrt{5}$.
b. The angle of rotation at $z = 2 + i$ is $\tan^{-1}(0.5)$.
c. The coefficient of magnification at $z = 1 + i$ is $2\sqrt{2}$.
d. The angle of rotation at $z = 1 + i$ is $\dfrac{\pi}{4}$.

Solution:

$$\begin{aligned} w = f(z) &= z^2 \\ \implies f'(z) &= 2z \\ \implies f'(2+i) &= 2(2+i) = 4+2i. \end{aligned}$$

(a.) Coefficient of magnification at $z = 2+i$ is
$|f'(2+i)| = |4+2i| = 2\sqrt{5}$.
(b) Angle of rotation at $z = 2+i$ is

$$ampf'(2+i) = (4+2i) = \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(0.5).$$

and $f'(1+i) = 2(1+i) = 2+2i$
(c) The coefficient of magnification at $z = 1+i$ is
$|f'(1+i)| = |2+2i| = \sqrt{4+4} = 2\sqrt{2}$
(d) The angle of rotation at $z = 1+i$ is

$$amp.f'(1+i) = 2+2i = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}.$$

Thanks !!!