Mathematics-II (Complex Variable)
Lecture Notes
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by

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Example

If $w = \phi + i\psi$ represents the complex potential for an electric field and

$$\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

determine the function ϕ.

Solution: We have, $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ so that

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2+y^2).1 - x.(2x)}{(x^2+y^2)^2} = 2x + \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y + \frac{-x.(2y)}{(x^2+y^2)^2} = -2y + \frac{-2xy}{(x^2+y^2)^2}$$

We know that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

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Using $C - R$ equations $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $\quad and \quad \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$

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Putting the values of $\dfrac{\partial \psi}{\partial x}$ and $\dfrac{\partial \psi}{\partial y}$,we get

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The R.H.S. is an exact differential equation of the form $Mdx + Ndy$. Hence its solution is

$$d\phi = \int \left(-2y + \frac{-2xy}{(x^2+y^2)^2}\right)dx \implies \phi = -2xy + \frac{y}{(x^2+y^2)} + c$$

Harmonic Conjugate ...

Dr. G.K. Prajapati

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Introduction

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Example

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic function and $u = -r^3 \sin 3\theta$. then construct the corresponding analytic function $f(z)$ in terms of z.

Solution: We have $u = -r^3 \sin 3\theta$. Then

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta \text{ and } \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

We know that

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

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Using $C-R$ equations in polar form
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad and \qquad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

$$dv=-\frac{1}{r}\frac{\partial u}{\partial \theta}dr+r\frac{\partial u}{\partial r}d\theta$$

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$$dv = -\frac{1}{r}(-3r^3\cos 3\theta)dr + r(-3r^2\sin 3\theta)d\theta$$

$$dv = (3r^2\cos 3\theta)dr - (3r^3\sin 3\theta)d\theta$$

The R.H.S. is an exact differential equation of the form $Mdr + Nd\theta$. Hence its solution is

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Now,

$$f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic = ir^3(\cos 3\theta + i \sin 3\theta) + ic$$

$$f(z) = ir^3 e^{3i\theta} + ic \implies f(z) = i\left(re^{i\theta}\right)^3 + ic = iz^3 + ic$$

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Example

If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.

Solution: $u + iv = f(z) \implies iu - v = if(z)$
Adding these, $(u - v) + i(u + v) = (1 + i)f(z)$ Let

$$U + iV = (1 + i)f(z) \text{ where } U = u - v \text{and } V = u + v$$

$$F(z) = (1 + i)f(z)$$

$$U = u - v = (x - y)(x^2 + 4xy + y^2) = x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 \text{and } \frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

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We know that

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

Using $C-R$ equations $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad and \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

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Putting the values of $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, we get

$$dV = -(3x^2 - 6xy - 3y^2)dx + (3x^2 + 6xy - 3y^2)dy$$

The R.H.S. is an exact differential equation of the form $Mdx + Ndy$. Hence its solution is

$$V = -\int (3x^2 - 6xy - 3y^2)dx + \int (-3y^2)dy \implies V = -x^3 + 3x^2y + 3xy^2 - y^3 + c$$

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Now,

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\begin{aligned}
F(z) &= U + iV \\
&= (x^3 + 3x^2y - 3xy^2 - y^3) + i(-x^3 + 3x^2y + 3xy^2 - y^3) \\
&= (1-i)x^3 + (1+i)3x^2y - (1-i)3xy^2 - (1+i)y^3 + ic \\
&= (1-i)x^3 + i(1-i)3x^2y - (1-i)3xy^2 - i(1-i)y^3 + i \\
&= (1-i)[x^3 + 3ix^2y - 3xy^2 - iy^3] + ic \\
&= (1-i)(x+iy)^3 + ic \\
&= (1-i)z^3 + ic
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Thus

$$(1+i)f(z) = (1-i)z^3 + ic,$$

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Thanks !!!