Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

A function $f(z)$ is said to be **analytic** at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.
A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.
The point at which the function is not differentiable is called a **singular point** of the function.
An analytic function is also known as **holomorphic**, **regular**, monogenic.

Definition

Entire Function: A function which is analytic everywhere (for all z in the complex plane) is known as an entire function.

Example

1. Polynomials rational functions are entire.
2. $|z|^2$ is differentiable only at $z = 0$. So it is no where analytic.

Remark

1. *An entire is always analytic, differentiable and continuous function. But converse is not true.*
2. *Analytic function is always differentiable and continuous. But converse is not true.*
3. *A differentiable function is always continuous. But converse is not true*

Theorem

The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{provided} \quad \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \quad \text{and} \quad \frac{\partial v}{\partial x} \text{ exists.}$$

Definition

Cauchy Riemann equations: The equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is known as Cauchy Riemann equations.

Theorem

The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ *and* $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

2. $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y},$ *and* $\dfrac{\partial v}{\partial x}$ *are continuous functions of x and y in region R.*

Example

Show that the function $e^x(\cos y + i \sin y)$ is an analytic function, find its derivative.

Solution: Let $e^x(\cos y + i \sin y) = u + iv$.
So, $e^x \cos y = u$ and $e^x \sin y = v$ then

$$\frac{\partial u}{\partial x} = e^x \cos y, \ \frac{\partial u}{\partial y} = -e^x \sin y, \ \frac{\partial v}{\partial x} = e^x \sin y, \quad \text{and} \quad \frac{\partial v}{\partial y} = e^x \cos y$$

Here we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus are $C - R$ equations and are satisfied and the partial derivatives are continuous. Hence, $e^x(\cos y + i \sin y)$ is analytic.

The derivative of the function $e^x(\cos y + i\sin y)$ is

$$f'(z) = u' + iv' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^x.e^{iy} = e^{x+iy} = e^z$$

Example

Discuss the analyticity of the function $f(z) = |z|^2$.

Solution:

$f(z) = |z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$

$$f(z) = x^2 + y^2 = u + iv \implies u = x^2 + y^2, v = 0$$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0} \frac{k^2}{k} = 0$$

Also

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,0) - v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,0+k) - v(0,0)}{k} = 0$$

Thus the $C-R$ equations $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ are satisfied and the partial derivatives are continuous. Hence, $f(z)=|z|^2$ is analytic at origin.

Example

Show that the function $f(z) = u + iv$, where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0 \\ 0, \qquad z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at $z = 0$. Is the function analytic at $z = 0$? Justify your answer.

Solution: $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = u + iv$

$$u = \frac{x^3 - y^3}{x^2+y^2}, \qquad and \qquad v = \frac{x^3 + y^3}{x^2+y^2}$$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0} \frac{-k^3/k^2}{k} = -1$$

Also

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,0) - v(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,0+k) - v(0,0)}{k} = \lim_{k \to 0} \frac{k^3/k^2}{k} = 1$$

Thus the $C - R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied. Again for derivatives

$$f'(0) = \lim_{z \to 0} \frac{f(z+0) - f(0)}{z} =$$

$$\lim_{z \to 0} \left[\frac{\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} - 0}{x+iy} \right] =$$

$$\lim_{z \to 0} \left[\frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)} \right]$$

Now let $z \to 0$ along $y = mx$, then

$$= \lim_{x\to 0}\left[\frac{x^3(1+i)-(mx)^3(1-i)}{(x^2+(mx)^2)(x+i(mx))}\right] =$$
$$\lim_{x\to 0}\left[\frac{(1+i)-(m)^3(1-i)}{(1+(m)^2)(1+im)}\right] = \left[\frac{(1+i)-(m)^3(1-i)}{(1+(m)^2)(1+im)}\right]$$

Which depends on the value of m. So for different paths we get different values of $\dfrac{df}{dz}$. In such a case, the function is not differentiable at $z = 0$. Hence given function is not analytic at $z = 0$.

Definition

C-R EQUATIONS IN POLAR FORM: The $C - R$ equations in polar form is

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Exercise

Determine which of the following functions are analytic:

(1.) $x^2 + iy^2$ **Ans.** Analytic at all points $y = x$

(2.) $2xy + i(x^2y^2)$ **Ans.** Not analytic

(3.) $\sin x \cosh y + i \cos x \sinh y$ **Ans.** Yes, analytic

(4.) Show the function of $\bar{z}$ is not analytic any where.

(5.) Discuss the analyticity of the function

$$f(z) = \begin{cases} \dfrac{x^2y(y - ix)}{x^4 + y^2}, z \neq 0 \\ 0, \qquad z = 0 \end{cases} \quad \text{at } z = 0.$$

Thanks !!!