Mathematics-II (Differential Equations)
Lecture Notes
April 20, 2020

by

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FUNCTIONS OF A COMPLEX VARIABLE: $f(z)$ is a function of a complex variable z and is denoted by w.

$$w = f(z)$$
$$w = u + iv$$

where u and v are the real and imaginary parts of $f(z)$. **NEIGHBORHOOD OF Z_0** :Let z_0 is a point in the complex plane and let z be any positive number, then the set of points z such that

$$|z - z_0| < \epsilon$$

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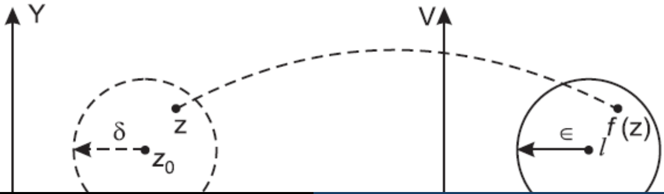
LIMIT: Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\epsilon > 0$, however small there exists a real number $\delta > 0$, such that

$$|f(z) - l| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

i.e. for every $z \neq z_0$ in δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the ϵ-disc of w-plane.

In symbolic form, $\lim_{z \to z_0} f(z) = l$.



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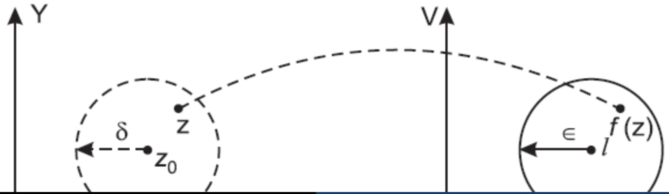

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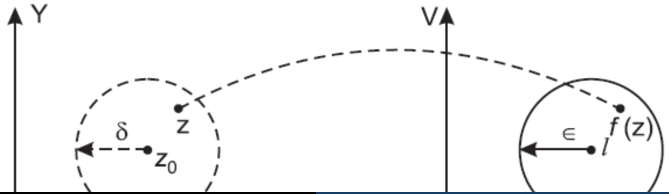

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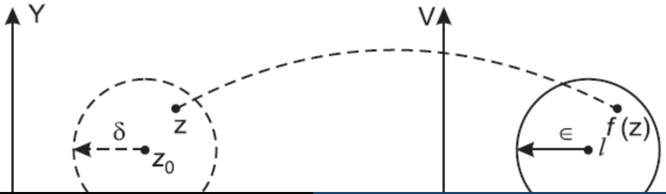

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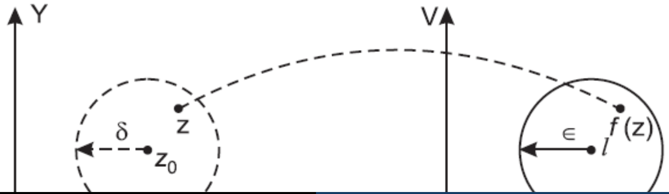
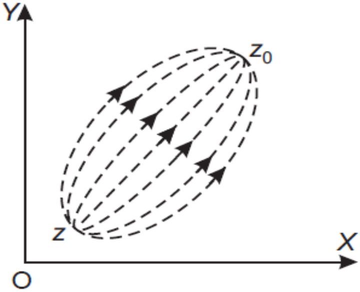


Note: (I) δ usually depends upon ϵ.
(II) $z \to z_0$ implies that z approaches z_0 along any path. The limits must be independent of the manner in which z approaches z_0 If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.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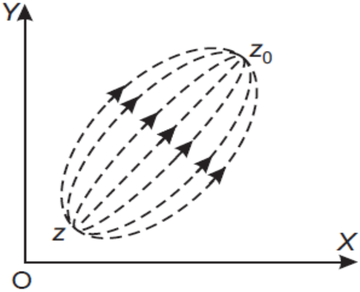
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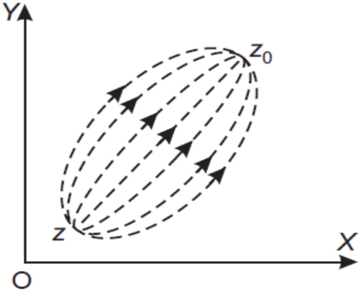
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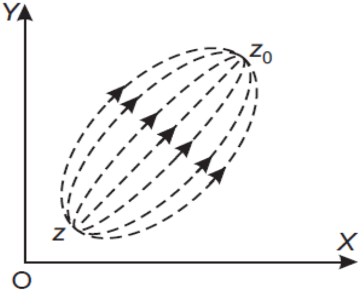
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Example

Prove that $$\lim_{z\to 1-i}\frac{z^2+4z+3}{z+1}=4-i$$

Solution:

$$\lim_{z\to 1-i}\frac{(z+1)(z+3)}{z+1}=\lim_{z\to 1-i}(z+3)=(1-i)+3=4-i$$

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Solution:

$$\lim_{z \to 1-i} \frac{\cancel{(z+1)}(z+3)}{\cancel{z+1}} = \lim_{z \to 1-i} (z+3) = (1-i)+3 = 4-i$$

Example

Show that $\lim_{z\to 0} \frac{z}{|z|}$ does not exist.

Solution: $\lim_{z\to 0} \frac{z}{|z|} = \lim_{(x,y)\to(0,0)} \frac{x+iy}{\sqrt{x^2+y^2}}$

Let $y = mx$,

$= \lim_{x\to 0} \frac{x+imx}{\sqrt{x^2+(mx)^2}} = \lim_{x\to 0} \frac{1+im}{\sqrt{1+(m)^2}} = \frac{1+im}{\sqrt{1+m^2}}$

The value of $\frac{1+im}{\sqrt{1+m^2}}$ are different for different value of m.

Hence the limit does not exist.

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Again **Case-2.** $\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{(x,y)\to(0,0)}\frac{x+iy}{x-iy} =$

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Complex function ...

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Exercise

Show that the limit does not exist

1. $\lim\limits_{z\to 0} \dfrac{Im(z)^3}{Re(z)^3}$ 2. $\lim\limits_{z\to 0} \dfrac{z}{(\bar{z})^2}$ 3. $\lim\limits_{z\to 0} \dfrac{Re(z)^2}{Im(z)}$

Find the limit of the following

5. $\lim\limits_{z\to 0} \dfrac{Re(z)^2}{|z|}$ **Ans.** 0 6. $\lim\limits_{z\to 1+i} \dfrac{2z^3}{(Im(z))^2}$ **Ans.** $2(-1+i)$

7. $\lim\limits_{z\to 0} \dfrac{z^2+6z+3}{z^2+2z+2}$ **Ans.** $3/2$.

Continuity: The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number ϵ, we can find a number δ such that $|f(z) - f(z_0)| < \epsilon$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to 0} f(z) = f(z_0)$$

Example

Examine the continuity of the function

$$f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, z \neq i \\ 0, \qquad\qquad z = i \end{cases}$$

at $z = i$

Solution:

$$\lim_{z \to i} f(z) = \lim_{z \to i} \frac{z^3 - iz^2 + z - i}{z - i} = \lim_{z \to i} \frac{z^2(z-i) + 1(z-i)}{z - i} = \lim_{z \to i} \frac{(z^2+1)(z-i)}{z-i}$$

$$= \lim_{z \to i}(z^2 + 1) = 0$$

Also, we have $f(i) = 0$. Thus

$$= \lim_{z \to i} f(z) = f(i)$$

Hence $f(z)$ is continuous at $z = i$.

Solution:

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$$\lim_{z\to i} f(z) = \lim_{z\to i}\frac{z^3 - iz^2 + z - i}{z-i} = \lim_{z\to i}\frac{z^2(z-i)+1(z-i)}{z-i} = \lim_{z\to i}\frac{(z^2+1)(z-i)}{z-i}$$

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Show that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{Re(z)}{z}, z \neq 0 \\ 0, \qquad\qquad z = 0 \end{cases}$$

is not continuous at $z = 0$.

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Exercise

Examine the continuity of the following functions

(1.) $$f(z) = \begin{cases} \dfrac{Im(z)}{|z|}, z \neq 0 \\ 0, \qquad z = 0 \end{cases} \quad \text{at } z = 0.$$

Ans. Not Continuous

(2.) $$f(z) = \frac{z^2 + 3z + 4}{z^2 + i} \quad \text{at } z = 1 - i$$

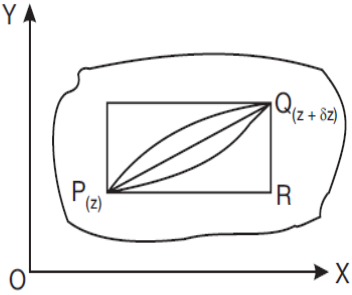
Ans. Continuous

DIFFERENTIABILITY:

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z \to 0$. Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path.


Y
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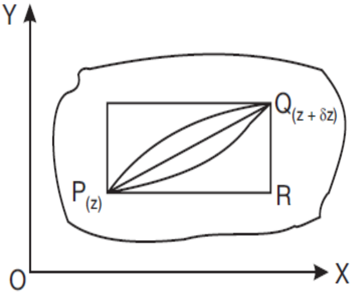
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Example

If $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, z \neq 0 \\ 0, \qquad z = 0 \end{cases}$ Then discuss $\dfrac{df}{dz}$ at $z = 0$.

Solution: If $z \to 0$ along radius vector $y = mx$.

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{\frac{x^3y(y-ix)}{x^6+y^2}-0}{x+iy}\right] =$$

$$\lim_{z\to 0}\left[\frac{-ix^3y(x+iy)}{(x^6+y^2)(x+iy)}\right]$$

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In different paths we get different values of $\frac{df}{dz}$ i.e. 0 and $-\frac{i}{2}$.In such a case, the function is not differentiable at $z = 0$.

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Example

Prove that the function $f(z) = |z|^2$ is continuous everywhere but no where differentiable except at the origin.

Thanks !!!