Mathematics-II (Differential Equations)
Lecture Notes
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by

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Laplace
equation in ...

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Introduction

Laplace Equation

Definition

A two dimensional Laplace equation is defined as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

and a three dimensional Laplace equation is defined as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2}$$

Laplace equation is also known as **potential equation**.

If the problems involves rectangular boundaries, we use the Laplace equation given by (1) and (2).

Laplace's Equation in plane polar coordinates

If the given boundary problem involves circular boundaries, we use Laplace's equation in polar coordinates (r, θ).

Example

Transform the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates (r, θ).

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Solution: If (x, y) be the Cartesian coordinate's of the point P whose polar coordinates are (r, θ), then

$$x = r\cos\theta \qquad and \qquad y = r\sin\theta \tag{3}$$

From (3)

$$r^2 = x^2 + y^2 \qquad and \qquad \theta = \tan^{-1}\frac{y}{x} \tag{4}$$

From (4)

$$2r\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \tag{5}$$

and

$$2r\frac{\partial r}{\partial y} = 2y \implies \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \tag{6}$$

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Also

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \left(-\frac{y}{x^2} \right) = -\frac{r \sin \theta}{r^2} = -\frac{\sin \theta}{r} \qquad (7)$$

and

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \left(\frac{1}{x} \right) = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r} \qquad (8)$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \implies \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

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Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right)\right]$$
$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial \theta} + \sin\theta\frac{\partial^2 u}{\partial \theta^2}\right)\right]$$

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Thus

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$
$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad (9)$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}$$

$$\implies \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} \implies \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

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Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2} + \cos\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right)\right]$$
$$+ \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{r}\left(-\sin\theta\frac{\partial u}{\partial\theta} + \cos\theta\frac{\partial^2 u}{\partial\theta^2}\right)\right]$$

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Thus

$$\frac{\partial^2 u}{\partial y^2} = \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} \quad (10)$$

Adding (9) and (10)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Hence Laplace equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

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Laplace's equation in cylindrical coordinates

If the given boundary problem involves cylindrical boundaries, we use Laplace's equation in cylindrical coordinates (r, θ, z).

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Example

Transform the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ into polar coordinates (r, θ, z).
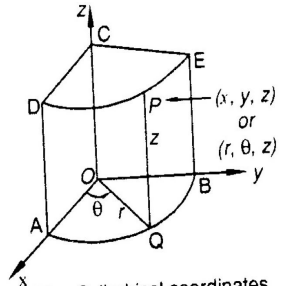
Solution:

Fig. Cylindrical coordinates

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If (x, y, z) be the Cartesian coordinate's of the point P whose cylindrical coordinates are (r, θ, z), then we know that

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$$x = r\cos\theta, \qquad y = r\sin\theta \qquad and \qquad z = z \qquad (11)$$

With $x = r\cos\theta, \quad and \quad y = r\sin\theta$, proceed as in the Example (2) and prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (12)$$

Adding $\dfrac{\partial^2 u}{\partial z^2}$ on both side of (12), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \qquad (13)$$

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Hence the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (14)$$

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Thanks !!!