Mathematics-II (Differential Equations)
Lecture Notes
April 16, 2020

by

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Example

Discuss D'Alembert's solution of one dimensional wave equation. or
Show that the general solution of the wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \text{ is } u(x,t) = \phi(x+ct) + \psi(x-ct),$$

where ϕ and ψ are arbitrary functions.

Solution: Given equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Let v and w be two new independent variables such that

$$w = x + ct \qquad and \qquad v = x - ct \tag{1}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial v}\frac{\partial v}{\partial x}$$

Using equation (1), we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial v} \tag{2}$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \implies \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial w} + \frac{\partial u}{\partial v}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \tag{3}$$

Again

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial u}{\partial v}\frac{\partial v}{\partial t}$$

Using equation (1), we have

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial w} - c\frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v}\right) \quad (4)$$

Thus

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) \implies \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial w} - \frac{\partial u}{\partial v}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2}\right)$$

$$\implies \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2}\right) \quad (5)$$

Using (3) and (5) reduces to

$$\frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} = \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \implies \frac{\partial^2 u}{\partial w \partial v} = 0 \tag{6}$$

$$\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) = 0 \tag{7}$$

Integrating (7) w.r.t. v, we get

$$\frac{\partial u}{\partial w} = F(w), \tag{8}$$

where F is an arbitrary function of w.
Integrating (8) w.r.t. w, we get

$$u = \int F(w)dw + \psi(v),$$

where ψ is an function of v. Then

$$u = \phi(w) + \psi(v), \text{ where } \phi(w) = \int F(w)dw$$

or

$$u = \phi(x + ct) + \psi(x - ct).$$

General solution of one-dimensional heat (diffusion) equation satisfying the given boundary and initial conditions

Consider one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$$

where $u(x,t)$ is the temperature of the bar. If both the ends of a bar of length a are at temperature zero and initial temperature is to be prescribed function $f(x)$ in the bar, then find the temperature at a subsequent time t. More precisely, the faces $x=0$ and $x=a$ of an infinite slab are maintained at zero temperature. Given that the temperature $u(x,t)=f(x)$ at $t=0$. Find the temperature at a subsequent time t.

Solution: Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \tag{9}$$

with boundary conditions $u(0, t) = 0$, $u(a, t) = 0$.
The initial condition is given by $u(x, 0) = f(x), \qquad 0 < x < a$
Let the given equation has the solution of the form $u(x, t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\frac{\partial u}{\partial t} = X(x)T'(t)$. Putting these values in given equation, we have

$$X''T = \frac{1}{k}XT' \implies \frac{X''}{X} = \frac{T'}{kT}, \tag{10}$$

Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T'}{kT} = \mu(constant) \implies X'' - \mu X = 0 \text{ and } T' - \mu kT = 0$$

These are ordinary differential equation of second order and first order with constant coefficient. Now to solve these two equations

$$X'' - \mu X = 0 \tag{11}$$

and

$$T' - \mu kT = 0. \tag{12}$$

Now three cases arises:

Case-I When $\mu = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution $u(x,t) = X(x)$T(t) becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = a_1 x + a_2$ becomes $0 = a_1.0 + a_2$ and $0 = a_1.a + a_2 \implies a_1 = 0 = a_2$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So we reject case-I, when $\mu = 0$.

Case-II When $\mu > 0$, we can take $\mu = \lambda^2 (say)$, then equations $X'' - \mu X = 0$ reduces to

$$X'' - \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 - \lambda^2) = 0 \implies m = \pm\lambda. \text{ Therefore its solution will be}$$
$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution $u(x,t)X(x)T(t)$ becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ becomes $0 = b_1 e^{\lambda.0} + b_2 e^{-\lambda.0}$ and $0 = b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies 0 = b_1 + b_2$ and $b_1 e^{\lambda a} + b_2 e^{-\lambda a} \implies b_1 = b_2 = 0$, so that $X(x) = 0$, which yields $u(x,t) = 0$. So again we reject case-II, when $\mu > 0$.

Case-III When $\mu < 0$, we can take $\mu = -\lambda^2(say)$, then first equations reduces to

$$X'' + \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i. \text{ Therefore its solution will be}$$
$$X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

Using boundary conditions $u(0,t) = 0 = u(a,t)$, the trial solution becomes

$$0 = X(0)T(t) \qquad \text{and} \qquad 0 = X(a)T(t).$$

Since $T(t) = 0 \implies u(x,t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ becomes $0 = c_1 \cos(\lambda.0) + c_2 \sin(\lambda.0)$ and
$0 = c_1 \cos(\lambda a) + c_2 \sin(\lambda a) \implies c_1 = 0$ and $c_2 \sin(\lambda a) = 0$

Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$\implies \sin \lambda a = 0 \implies \lambda a = n\pi \quad n = 1, 2, 3, ...$$

Thus $\lambda = \dfrac{n\pi}{a}, n = 1, 2, 3, ...$
Hence non-zero solution $X_n(x)$ are given by

$$X_n(x) = (c_2)_n \sin\left(\frac{n\pi x}{a}\right) \tag{13}$$

Now the solution corresponding to the equation $T' + \lambda^2 kT = 0$ is

$$\frac{T'}{T} = -\lambda^2 k \tag{14}$$

By integrating we get

$$\log T = -\lambda^2 kt + \log c_3 \implies T = c_3 e^{-\lambda^2 kt} \implies T = c_3 e^{-(n^2\pi^2/a^2)} \tag{15}$$

Hence solution is $T_n(t) = D_n e^{-C_n^2 t}$, where $C_n = (n^2\pi^2 k/a^2)$ and $D_n = c_3$ are new arbitrary constants.
The general solution is

$$u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-C_n^2 t}, \tag{16}$$

where $E_n = (c_2)_n D_n$ is another new arbitrary constants.

Substituting $t = 0$ in (16) and using initial condition $u(x, 0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) \quad (17)$$

Which are Fourier sin series of expansion $f(x)$. Accordingly we get

$$E_n = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a} dx \quad (18)$$

Hence the required solution is given by the equation (16) and E_n given by the equation (18).

Thanks !!!