

Lok Nayak Jai Prakash Institute of Technology Chapra, Bihar-841302

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Mathematics-II (Differential Equations) Lecture Notes April 16, 2020

by

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Example

Discuss D'Alembert's solution of one dimensional wave equation. or

Show that the general solution of the wave equation

$$
c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}
$$
 is $u(x,t) = \phi(x+ct) + \psi(x-ct)$,

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where ϕ and ψ are arbitrary functions.

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Solution: Given equation is

$$
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
$$

Let v and w be two new independent variables such that

$$
w = x + ct \qquad and \qquad v = x - ct \tag{1}
$$

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Now

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial v} \frac{\partial v}{\partial x}
$$

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Using equation [\(1\)](#page-2-0), we have

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial w} + \frac{\partial}{\partial v} \qquad (2)
$$
\nThus\n
$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \implies \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial v} \right) \left(\frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} \right)
$$
\n
$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2 \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \qquad (3)
$$

Again

$$
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w} \frac{\partial w}{\partial t} + \frac{\partial u}{\partial v} \frac{\partial v}{\partial t}
$$

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Using equation [\(1\)](#page-2-0), we have

$$
\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial w} - c \frac{\partial u}{\partial v} \qquad So \quad that \qquad \frac{\partial}{\partial t} = c \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v} \right) \tag{4}
$$

Thus

$$
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) \implies \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v} \right) \left(\frac{\partial u}{\partial w} - \frac{\partial u}{\partial v} \right)
$$

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial w^2} - 2 \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \right)
$$

$$
\implies \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial w^2} - 2 \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \right)
$$
(5)

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Using (3) and (5) reduces to
\n
$$
\frac{\partial^2 u}{\partial w^2} + 2 \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} = \frac{\partial^2 u}{\partial w^2} - 2 \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial v^2} \implies \frac{\partial^2 u}{\partial w \partial v} = 0
$$
\n(6)\n
$$
\frac{\partial}{\partial v} \left(\frac{\partial u}{\partial w} \right) = 0
$$
\n(7)

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Integrating (7) w.r.t. v , we get

$$
\frac{\partial u}{\partial w} = F(w),\tag{8}
$$

where F is an arbitrary function of w . Integrating (8) w.r.t. w , we get

$$
u = \int F(w)dw + \psi(v),
$$

where ψ is an function of v . Then

$$
u = \phi(w) + \psi(v), \text{ where } \phi(w) = \int F(w) dw
$$

or

$$
u = \phi(x + ct) + \psi(x - ct).
$$

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General solution of one-dimensional heat (diffusion) equation satisfying the given boundary and initial conditions

Consider one-dimensional heat equation

$$
\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t},
$$

where $u(x, t)$ is the temperature of the bar. If both the ends of a bar of length a are at temperature zero and initial temperature is to be prescribed function $f(x)$ in the bar, then find the temperature at a subsequent time t . More precisely, the faces $x = 0$ and $x = a$ of an infinite slab are maintained at zero temperature. Given that the temperature $u(x,t) = f(x)$ a[t](#page-0-0) $t = 0$ $t = 0$. Find the temperature at a s[ub](#page-6-1)[se](#page-8-0)[q](#page-6-1)[ue](#page-7-0)[n](#page-8-0)t t[im](#page-16-0)[e](#page-0-0) t_{eff} [.](#page-16-0)

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Solution: Given that

$$
\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t},\tag{9}
$$

with boundary conditions $u(0, t) = 0$, $u(a, t) = 0$. The initial condition is given by $u(x, 0) = f(x)$, $0 < x < a$ Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where X is function of x alone and T is function of t alone. Now $\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\dfrac{\partial u}{\partial t} = X(x) T'(t).$ Putting these values in given equation, we have 1 X'' T \prime

$$
X''T = \frac{1}{k}XT' \implies \frac{X''}{X} = \frac{T'}{kT},\tag{10}
$$

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Since x and t are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$
\frac{X''}{X} = \frac{T'}{kT} = \mu(constant) \implies X'' - \mu X = 0 \text{ and}
$$

$$
T' - \mu kT = 0
$$

These are ordinary differential equation of second order and first order with constant coefficient. Now to solve these two equations

$$
X'' - \mu X = 0 \tag{11}
$$

and

$$
T' - \mu kT = 0. \tag{12}
$$

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Now three cases arises:

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Case-I When $\mu = 0$, then both equations reduces to

$$
X'' = 0 \implies X = a_1 x + a_2
$$

Using boundary conditions $u(0, t) = 0 = u(a, t)$, the trial solution $u(x, t) = X(x)T(t)$ becomes

 $0 = X(0)T(t)$ and $0 = X(a)T(t)$.

Since $T(t) = 0 \implies u(x, t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = a_1x + a_2$ becomes $0 = a_1.0 + a_2$ and $0 = a_1.a + a_2 \implies a_1 = 0 = a_2$, so that $X(x) = 0$, which yields $u(x, t) = 0$. So we reject case-I, when $\mu = 0.$

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Case-II When $\mu > 0$, we can take $\mu = \lambda^2 (say)$, then equations $X'' - \mu X = 0$ reduces to

 $X''-\lambda^2 X=0\implies$ the auxiliary equation is $(m^2 - \lambda^2) = 0 \implies m = \pm \lambda$. Therefore its solution will be $X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$

Using boundary conditions $u(0, t) = 0 = u(a, t)$, the trial solution $u(x,t)X(x)T(t)$ becomes

 $0 = X(0)T(t)$ and $0 = X(a)T(t)$.

Since $T(t) = 0 \implies u(x, t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X=b_1e^{\lambda x}+b_2e^{-\lambda x}$ becomes $0=b_1e^{\lambda.0}+b_2e^{-\lambda.0}$ and $0=b_1e^{\lambda a}+b_2e^{-\lambda a} \implies 0=b_1+b_2$ and $b_1e^{\lambda a}+b_2e^{-\lambda a} \implies b_1=b_2=0$, so that $X(x)=0$, which yi[e](#page-1-0)lds $u(x,t) = 0$ $u(x,t) = 0$. So again we reject [ca](#page-10-0)[se](#page-12-0)[-I](#page-10-0)[I,](#page-11-0) [w](#page-12-0)[h](#page-0-0)e[n](#page-16-0) $\mu \geq 0$.

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Case-III When $\mu < 0$, we can take $\mu = -\lambda^2 (say)$, then first equations reduces to

 $X'' + \lambda^2 X = 0 \implies$ the auxiliary equation is $(m^2 + \lambda^2) = 0 \implies m = \pm \lambda i$. Therefore its solution will be $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$

Using boundary conditions $u(0, t) = 0 = u(a, t)$, the trial solution becomes

 $0 = X(0)T(t)$ and $0 = X(a)T(t)$.

Since $T(t) = 0 \implies u(x, t) = 0$, so we suppose that $T(t) \neq 0$. Then we have $X(0) = 0$ and $X(a) = 0$. Now using these boundary conditions, the solution $X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$ becomes $0 = c_1 \cos(\lambda.0) + c_2 \sin(\lambda.0)$ and $0 = c_1 \cos(\lambda a) + c_2 \sin(\lambda a) \implies c_1 = 0$ $0 = c_1 \cos(\lambda a) + c_2 \sin(\lambda a) \implies c_1 = 0$ [a](#page-1-0)[n](#page-11-0)[d](#page-12-0) $c_2 \sin(\lambda a) = 0$ $c_2 \sin(\lambda a) = 0$ Dr. G.K. Prajapati LNJPIT, Chapra [Classification of Partial Differential ...](#page-0-0)

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Now for non-trivial solution of given wave equation, we can not take $c_2 = 0$

$$
\implies \sin \lambda a = 0 \implies \lambda a = n\pi \quad n = 1, 2, 3, \dots
$$

Thus
$$
\lambda = \frac{n\pi}{a}
$$
, $n = 1, 2, 3, ...$
Hence non-zero solution $X_n(x)$ are given by

$$
X_n(x) = (c_2)_n \sin\left(\frac{n\pi x}{a}\right) \tag{13}
$$

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By integrating we get

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Now the solution corresponding to the equation $T'+\lambda^2 kT=0$ is \prime

$$
\frac{T'}{T} = -\lambda^2 k \tag{14}
$$

 $\log T = -\lambda^2 k t + \log c_3 \implies T = c_3 e^{-\lambda^2 k t} \implies T = c_3 e^{-(n^2 \pi^2/a^2)}$ (15) Hence solution is $T_n(t) = D_n e^{-C_n^2 t}$, where $C_n = (n^2 \pi^2 k / a^2)$ and $D_n = c_3$ are new arbitrary constants. The general solution is

$$
u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-C_n^2 t},\tag{16}
$$

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where $E_n = (c_2)_n D_n$ $E_n = (c_2)_n D_n$ $E_n = (c_2)_n D_n$ is another new [arb](#page-13-0)[itr](#page-15-0)[a](#page-0-0)[ry](#page-14-0) [c](#page-15-0)[o](#page-0-0)n[st](#page-16-0)an[ts.](#page-16-0)

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Substituting $t = 0$ in [\(16\)](#page-14-1) and using initial condition $u(x, 0) = f(x)$, we get

$$
f(x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)
$$
 (17)

Which are Fourier sin series of expansion $f(x)$. Accordingly we get

$$
E_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n \pi x}{a} dx \tag{18}
$$

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Hence the required solution is given by the equation [\(16\)](#page-14-1) and E_n given by the equation [\(18\)](#page-15-1).

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