# Mathematics-II (Differential Equations)
## Lecture Notes
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by

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## METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

**Notations:** Let $u(x,t)$ is a function of two variable $x$ and $t$. We use the following notations:

$$\frac{\partial u}{\partial x} = u_x = u_x(x,t), \quad \frac{\partial u}{\partial t} = u_t = u_t(x,t),$$
$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = u_x(\pi,t), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = u_t(x,0)$$

**Example**

Solve the boundary value problem $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, if $u(0, y) = 8e^{-3y}$.

**Solution:** Given that

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \tag{1}$$

with boundary condition $u(0, y) = 8e^{-3y}$.

Let the given equation has the solution of the form $u(x,y) = X(x)Y(y)$, where $X$ is function of $x$ alone and $Y$ is function of $y$ alone. Now $\frac{\partial u}{\partial x} = X'(x)Y(y)$ and $\frac{\partial u}{\partial y} = X(x)Y'(y)$. Putting these values in given equation, we have

$$X'Y = 4XY' \implies \frac{X'}{4X} = \frac{Y'}{Y}, \quad (2)$$

Since $x$ and $y$ are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X'}{4X} = \frac{Y'}{Y} = k(constant) \implies X' - 4kX = 0 \text{ and } Y' - kY = 0$$

These are ordinary differential equation of first order first degree. Therefore its solutions will be

$$\frac{X'}{X} = 4k \implies \log X = 4kx + \log c_1 \implies \frac{X}{c_1} = 4kx \implies X = c_1 e^{4kx}$$

Similarly solution corresponding to $Y' - kY = 0$ , we get $Y = c_2 e^{ky}$. Substituting the values of $X$ and $Y$ in the trail solution $u(x, y) = X(x)Y(y)$ i.e.

$$u(x, y) = c_1 e^{4kx}.c_2 e^{ky} \implies u(x, y) = Ce^{4kx+ky},$$

where $C = c_1 c_2$ is another arbitrary constant.

Now putting $x = 0$ and using boundary condition $u(0, y) = 8e^{-3y}$, we have

$$u(0, y) = Ce^{4k.0+ky} \implies 8e^{-3y} = Ce^{ky}$$

Thus we have $C = 8$ and $k = -3$. Thus the required solution will be $u(x, y) = 8e^{-12x-3y}$.

**Example**

Using the method of separation of variable, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x, 0) = 6e^{-3x}$.

**Solution:** Given that

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \tag{3}$$

with boundary condition $u(x, 0) = 6e^{-3x}$.

Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where $X$ is function of $x$ alone and $T$ is function of $t$ alone. Now $\frac{\partial u}{\partial x} = X'(x)T(t)$ and $\frac{\partial u}{\partial t} = X(x)T'(t)$. Putting these values in given equation, we have

$$X'T = 2XT' + XT \implies X'T = X(2T' + T) \implies \frac{X'}{X} = 2\frac{T'}{T} + 1, \quad (4)$$

Since $x$ and $t$ are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X'}{X} = 2\left(\frac{T'}{T}\right) + 1 = k(constant) \implies X' - kX = 0 \text{ and } 2T' + T - kT = 0$$

These are ordinary differential equation of first order first degree. Therefore its solutions will be

$$X' - kX = 0 \implies \frac{X'}{X} = k \implies \log X = kx + \log c_1 \implies \frac{X}{c_1} = kx \implies X = c_1 e^{kx}$$

Now, solution corresponding to

$$2T' + T - kT = 0 \implies 2T' = T(k-1) \implies 2\frac{T'}{T} = \frac{(k-1)}{2},$$

we get $T = c_2 e^{\frac{(k-1)}{2}t}$. Substituting the values of $X$ and $T$ in the trail solution $u(x,t) = X(x)T(t)$ i.e.

$$u(x,t) = c_1 e^{kx}.c_2 e^{\frac{(k-1)t}{2}} \implies u(x,t) = Ce^{kx+\frac{(k-1)t}{2}},$$

where $C = c_1 c_2$ is another arbitrary constant.

Now putting $t = 0$ and using boundary condition $u(x, 0) = 6e^{-3x}$, we have

$$u(x, 0) = Ce^{kx + \frac{(k-1).0}{2}} \implies 6e^{-3x} = Ce^{kx}$$

Thus we have $C = 6$ and $k = -3$. Thus the required solution will be $u(x, t) = 6e^{-3x-2y}$.

## General solution of one-dimensional wave (vibrational) equation satisfying the given boundary and initial conditions

Consider one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with boundary conditions $u(0,t) = 0$ and $u(a,t) = 0, \quad \forall t.$

**Solution:** Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{5}$$

with boundary conditions $u(0,t) = 0$ and $u(a,t) = 0$.
Let the given equation has the solution of the form $u(x,t) = X(x)T(t)$, where $X$ is function of $x$ alone and $T$ is function of $t$ alone. Now $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ and $\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$. Putting these values in given equation, we have

$$X''T = \frac{1}{c^2}XT'' \implies \frac{X''}{X} = \frac{T''}{c^2T}, \tag{6}$$

Since $x$ and $t$ are independent variables, therefore above equation can only true if each side is equal to the same constant. i.e.

$$\frac{X''}{X} = \frac{T''}{c^2T} = k(constant) \implies X'' - kX = 0 \text{ and }$$
$$T'' - c^2kT = 0$$

These are ordinary differential equation of second order with constant coefficient. Now to solve these two equations $X'' - kX = 0$ and $T'' - c^2kT = 0$, three cases arises:

**Case-I** When $k = 0$, then both equations reduces to

$$X'' = 0 \implies X = a_1 x + a_2$$

and

$$T'' = 0 \implies T = a_3 t + a_4.$$

Thus the required solution is

$$u(x,t) = (a_1 x + a_2)(a_3 t + a_4). \tag{7}$$

**Case-II** When $k > 0$, we can take $k = \lambda^2 (say)$, then both equations reduces to

$$X'' - \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 - \lambda^2) = 0 \implies m = \pm\lambda. \text{ Therefore its solution will be}$$
$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$$

and

$$T'' - c^2 \lambda^2 T = 0 \implies T = b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}.$$

Thus the required solution is

$$u(x,t) = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})(b_3 e^{c\lambda t} + b_4 e^{-c\lambda t}). \tag{8}$$

**Case-III** When $k < 0$, we can take $k = -\lambda^2 (say)$, then both equations reduces to

$$X'' + \lambda^2 X = 0 \implies \text{the auxiliary equation is}$$
$$(m^2 + \lambda^2) = 0 \implies m = \pm\lambda i. \text{ Therefore its solution will be}$$
$$X = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

and

$$T'' + c^2\lambda^2 T = 0 \implies T = c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t).$$

Thus the required solution is

$$u(x,t) = (c_1 \cos(\lambda x) + c_2 \sin(\lambda x))(c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)). \tag{9}$$

Thus the equation (7), (8) and (9) are various possible solution of the given wave equation.

Thanks !!!