Mathematics-II (Differential Equations)
Lecture Notes
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by

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Classification of second order partial differential equations

Consider a general second order partial differential equation for a function of two variables x and y in the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$,$r = \dfrac{\partial^2 z}{\partial x^2}$,$s = \dfrac{\partial^2 z}{\partial x \partial y}$,$t = \dfrac{\partial^2 z}{\partial y^2}$. Also R, S and T are continuous functions of x and y only possessing partial derivatives defined in a domain D on the $(x - y)$ plan.

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Then the given equation is said to be

- Hyperbolic at a point (x, y) in domain D if $S^2 - 4RT > 0$
- Parabolic at a point (x, y) in domain D if $S^2 - 4RT = 0$
- Elliptic at a point (x, y) in domain D if $S^2 - 4RT < 0$

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Example

Classify the following partial differential equation

1. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$.
2. $2\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = 2.$
3. $(xy-1)r - 2(x^2y^2-1)s - (xy+1)t + xp + yq = 0$

Solution (1.) The given equation can be written as $r - t = 0$.
Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

We have $R = 1$, $S = 0$ and $T = -1$.Put these values in $S^2 - 4RT = (0)^2 - 4.(1)(-1) \implies S^2 - 4RT = 4 > 0$.
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Solution (2.) The given equation can be written as $2r + s + 3t - 2 = 0$. Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

We have $R = 2$, $S = 1$ and $T = 3$. Put these values in $S^2 - 4RT = (1)^2 - 4.(2)(3) \implies S^2 - 4RT = -23 < 0$. Therefore the given equation is elliptic.

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$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

We have $R = (xy - 1)$,$S = -2(x^2y^2 - 1)$ and $T = -(xy + 1)$. Put these values in

$$S^2 - 4RT = (-2(x^2y^2 - 1))^2 - 4.((xy - 1)).(-(xy + 1)) \implies 4(x^2y^2 - 1))^2 + 4.((x^2y^2 - 1)).$$

$$S^2 - 4RT = 4x^2y^2(x^2y^2 - 1).$$

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$$S^2 - 4RT = 4x^2y^2(x^2y^2 - 1).$$

Case-1: Either $x = 0$ or $y = 0$ or both $x = y = 0$. In this case $S^2 - 4RT = 0$, hence given equation is parabola.
Case-2: If $xy = \pm 1$, then in this case $S^2 - 4RT = 0$,hence given equation is parabola.
Case-3: If $x^2y^2 > 1$, then in this case $S^2 - 4RT > 0$,hence given equation is hyperbola.
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Example ((AKU-CE-II,2019))

Classify the partial differential equation
$\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 6u = 0.$

Solution: Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

We have $R = x$, $S = t$ and $T = 1$.Put these values in

$$S^2 - 4RT = (t)^2 - 4.(x).(1) \implies S^2 - 4RT = t^2 - 4x.$$

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Case-2: If $x < t^2/4$, then in this case $S^2 - 4RT > 0$,hence given equation is hyperbola.
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Example ((AKU-CE-II,2019))

The region in which the following partial differential equation $x^3\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 27\frac{\partial^2 u}{\partial y^2} + 5u = 0.$

Solution: Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

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Exercise

Classify the following PDE:

(1.) $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

(2.) $$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$$

(3.) $$xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$$

(4.) $$x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$$

Thanks !!!