Mathematics-II (Differential Equations)
Lecture Notes
April 11, 2020

by

Dr. G.K.Prajapati
Department of Applied Science and Humanities
LNJPIT, Chapra, Bihar-841302

Introduction

Case-III: When $\phi(x, y) = x^m y^n$.

Then

$$P.I. = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} = x^m y^n.$$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D/D' or D'/D as the case may be.

Case-III: When $\phi(x,y) = x^m y^n$.
Then

$$P.I. = \frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} = x^m y^n.$$

which is evaluated by expanding $[F(D,D')]^{-1}$ in ascending powers of D/D' or D'/D as the case may be.

Case-III: When $\phi(x, y) = x^m y^n$.
Then

$$P.I. = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} = x^m y^n.$$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D/D' or D'/D as the case may be.

Case-III: When $\phi(x, y) = x^m y^n$.
Then

$$P.I. = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} = x^m y^n.$$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D/D' or D'/D as the case may be.

Example

Solve the PDE $s + p - q = z + xy$.

The given equation can be rewritten as
$(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$.
The complementary function (C.F.) is

$$e^x f_1(y) + e^{-y} f_2(x), \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

Example

Solve the PDE $s + p - q = z + xy$.

The given equation can be rewritten as
$(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$.
The complementary function (C.F.) is

$$e^x f_1(y) + e^{-y} f_2(x), \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

Example

Solve the PDE $s + p - q = z + xy$.

The given equation can be rewritten as
$(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$.
The complementary function (C.F.) is

$e^x f_1(y) + e^{-y} f_2(x)$, where f_1 and f_2 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

Example

Solve the PDE $s+p-q=z+xy$.

The given equation can be rewritten as
$(DD'+D-D'-1)z=xy \implies (D-1)(D'+1)z=xy$.
The complementary function (C.F.) is

$$e^x f_1(y)+e^{-y} f_2(x), \text{ where } f_1 \text{ and } f_2 \text{ are arbitrary function.}$$

Now

$$P.I.=\frac{1}{F(D,D')}x^m y^n=\frac{1}{(D-1)(D'+1)}xy.$$

Example

Solve the PDE $s + p - q = z + xy$.

The given equation can be rewritten as
$(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$.
The complementary function (C.F.) is

$e^x f_1(y) + e^{-y} f_2(x)$, where f_1 and f_2 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

Example

Solve the PDE $s + p - q = z + xy$.

The given equation can be rewritten as
$(DD' + D - D' - 1)z = xy \implies (D-1)(D'+1)z = xy$.
The complementary function (C.F.) is

$e^x f_1(y) + e^{-y} f_2(x)$, where f_1 and f_2 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{(D-1)(D'+1)} xy.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[1 - D' + D'^2 - ...\right] xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[xy - D'(xy) + D'^2(xy) - ...\right].$$

$$= -\left[1 + D + D^2 + ...\right](xy - x).$$

$$= -\left[(xy - x) + D(xy - x) + D^2(xy - x) + ...\right].$$

$$= -\left[(xy - x) + (y - 1)\right].$$

$$= -xy + x - y + 1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1+D+D^2+...\right]\left[1-D'+D'^2-...\right]xy.$$

$$= -\left[1+D+D^2+...\right]\left[xy-D'(xy)+D'^2(xy)-...\right].$$

$$= -\left[1+D+D^2+...\right](xy-x).$$

$$= -\left[(xy-x)+D(xy-x)+D^2(xy-x)+...\right].$$

$$= -\left[(xy-x)+(y-1)\right].$$

$$= -xy+x-y+1.$$

Non-Homogeneous Linear Partial Differential ...

Dr. G.K. Prajapati

LNJPIT, Chapra

Introduction

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1+D+D^2+...\right]\left[1-D'+D'^2-...\right]xy.$$

$$= -\left[1+D+D^2+...\right]\left[xy-D'(xy)+D'^2(xy)-...\right].$$

$$= -\left[1+D+D^2+...\right](xy-x).$$

$$= -\left[(xy-x)+D(xy-x)+D^2(xy-x)+...\right].$$

$$= -\left[(xy-x)+(y-1)\right].$$

$$= -xy+x-y+1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1+D+D^2+...\right]\left[1-D'+D'^2-...\right]xy.$$

$$= -\left[1+D+D^2+...\right]\left[xy-D'(xy)+D'^2(xy)-...\right].$$

$$= -\left[1+D+D^2+...\right](xy-x).$$

$$= -\left[(xy-x)+D(xy-x)+D^2(xy-x)+...\right].$$

$$= -\left[(xy-x)+(y-1)\right].$$

$$= -xy+x-y+1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1+D+D^2+...\right]\left[1-D'+D'^2-...\right]xy.$$

$$= -\left[1+D+D^2+...\right]\left[xy-D'(xy)+D'^2(xy)-...\right].$$

$$= -\left[1+D+D^2+...\right](xy-x).$$

$$= -\left[(xy-x)+D(xy-x)+D^2(xy-x)+...\right].$$

$$= -\left[(xy-x)+(y-1)\right].$$

$$= -xy+x-y+1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[1 - D' + D'^2 - ...\right]xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[xy - D'(xy) + D'^2(xy) - ...\right].$$

$$= -\left[1 + D + D^2 + ...\right](xy - x).$$

$$= -\left[(xy - x) + D(xy - x) + D^2(xy - x) + ...\right].$$

$$= -\left[(xy - x) + (y - 1)\right].$$

$$= -xy + x - y + 1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[1 - D' + D'^2 - ...\right]xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[xy - D'(xy) + D'^2(xy) - ...\right].$$

$$= -\left[1 + D + D^2 + ...\right](xy - x).$$

$$= -\left[(xy - x) + D(xy - x) + D^2(xy - x) + ...\right].$$

$$= -\left[(xy - x) + (y - 1)\right].$$

$$= -xy + x - y + 1.$$

$$= -\frac{1}{(1-D)(1+D')}xy \implies -(1-D)^{-1}(1+D')^{-1}xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[1 - D' + D'^2 - ...\right]xy.$$

$$= -\left[1 + D + D^2 + ...\right]\left[xy - D'(xy) + D'^2(xy) - ...\right].$$

$$= -\left[1 + D + D^2 + ...\right](xy - x).$$

$$= -\left[(xy - x) + D\,(xy - x) + D^2\,(xy - x) + ...\right].$$

$$= -\left[(xy - x) + (y - 1)\right].$$

$$= -xy + x - y + 1.$$

Therefore the required solution is

$$z = e^x f_1(y) + e^{-y} f_2(x) - xy + x - y + 1.$$

Therefore the required solution is

$$z = e^x f_1(y) + e^{-y} f_2(x) - xy + x - y + 1.$$

Example

Solve the PDE $r - s + p = 1$.

The given equation can be rewritten as
$(D^2 - DD' + D)z = 1 \implies D(D - D' + 1)z = 1$.
The complementary function (C.F.) is

$f_1(y) + e^{-x} f_2(y + x)$, where f_1 and f_2 are arbitrary function.

Example

Solve the PDE $r - s + p = 1$.

The given equation can be rewritten as
$(D^2 - DD' + D)z = 1 \implies D(D - D' + 1)z = 1.$
The complementary function (C.F.) is

$f_1(y) + e^{-x} f_2(y + x)$, where f_1 and f_2 are arbitrary function.

Example

Solve the PDE $r - s + p = 1$.

The given equation can be rewritten as
$(D^2 - DD' + D)z = 1 \implies D(D - D' + 1)z = 1$.
The complementary function (C.F.) is

$f_1(y) + e^{-x} f_2(y + x)$, where f_1 and f_2 are arbitrary function.

Example

Solve the PDE $r - s + p = 1.$

The given equation can be rewritten as
$(D^2 - DD' + D)z = 1 \implies D(D - D' + 1)z = 1.$
The complementary function (C.F.) is

$f_1(y) + e^{-x} f_2(y + x)$, where f_1 and f_2 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D,D')}x^m y^n = \frac{1}{D(D-D'+1)}.1$$

$$= \frac{1}{D}(1+D-D')^{-1}.1 \implies$$
$$\frac{1}{D}\left[1-(D-D')+(D-D')^2-...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x+y) + x.$$

Now

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{D(D-D'+1)}.1$$

$$= \frac{1}{D}(1+D-D')^{-1}.1 \implies$$

$$\frac{1}{D}\left[1-(D-D')+(D-D')^2-...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x+y) + x.$$

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{D(D - D' + 1)}.1$$

$$= \frac{1}{D}(1 + D - D')^{-1}.1 \implies$$

$$\frac{1}{D}\left[1 - (D - D') + (D - D')^2 - ...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x + y) + x.$$

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{D(D - D' + 1)}.1$$

$$= \frac{1}{D}(1 + D - D')^{-1}.1 \implies$$
$$\frac{1}{D}\left[1 - (D - D') + (D - D')^2 - ...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x + y) + x.$$

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{D(D - D' + 1)}.1$$

$$= \frac{1}{D}(1 + D - D')^{-1}.1 \implies$$
$$\frac{1}{D}\left[1 - (D - D') + (D - D')^2 - ...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x + y) + x.$$

Now

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{D(D - D' + 1)}.1$$

$$= \frac{1}{D}(1 + D - D')^{-1}.1 \implies$$

$$\frac{1}{D}\left[1 - (D - D') + (D - D')^2 - ...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x + y) + x.$$

Now

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{D(D-D'+1)}.1$$

$$= \frac{1}{D}(1+D-D')^{-1}.1 \implies$$
$$\frac{1}{D}\left[1-(D-D')+(D-D')^2-...\right].1$$

$$= \frac{1}{D}.1 \implies = x$$

Therefore the required solution is

$$z = f_1(y) + e^{-x} f_2(x+y) + x.$$

Example

Solve the PDE $D(D+D'-1)(D+3D'-2)z = x^2 - 4xy + 2y^2$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x)$, where f_1,f_2 and f_3 are arbitrary function.

Example

Solve the PDE $D(D+D'-1)(D+3D'-2)z = x^2-4xy+2y^2$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x)$, where f_1,f_2 and f_3 are arbitrary function.

Example

Solve the PDE $D(D+D'-1)(D+3D'-2)z = x^2 - 4xy + 2y^2$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x)$, where f_1,f_2 and f_3 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D,D')}(x^2 - 4xy + 2y^2) =$$

$$\frac{1}{2D}(1-(D+D')^{-1})\left\{1-\frac{D+3D'}{2}\right\}^{-1}(x^2-4xy+2y^2)$$

$$=$$

$$\frac{1}{2D}\{1+(D+D')+(D+D')^2+...\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3I}{2}\right.\right.$$

$$4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\right.$$

$$\left.\frac{(D+D')(D+3D')}{2}...\right\}(x^2-4xy+2y^2$$

Now

$$P.I. = \frac{1}{F(D,D')}(x^2 - 4xy + 2y^2) =$$

$$\frac{1}{2D}(1-(D+D')^{-1})\left\{1-\frac{D+3D'}{2}\right\}^{-1}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\{1+(D+D')+(D+D')^2+\ldots\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3I}{2}\right.\right.$$

$$\left. 4xy+2y^2\right)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\right.$$

$$\left.\frac{(D+D')(D+3D')}{2}\ldots\right\}(x^2-4xy+2y^2$$

Now

$$P.I. = \frac{1}{F(D,D')}(x^2 - 4xy + 2y^2) =$$

$$\frac{1}{2D}(1-(D+D')^{-1})\left\{1 - \frac{D+3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$=$$

$$\frac{1}{2D}\left\{1 + (D+D') + (D+D')^2 + ...\right\}\left\{1 + \frac{D+3D'}{2} + \left(\frac{D+3I}{2}\right.\right.$$

$$4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1 + (D+D') + (D+D')^2 + \frac{D+3D'}{2} + \left(\frac{D+3D'}{2}\right)^2 + \right.$$

$$\left.\frac{(D+D')(D+3D')}{2}...\right\}(x^2 - 4xy + 2y^2$$

Now

$$P.I. = \frac{1}{F(D, D')}(x^2 - 4xy + 2y^2) = \frac{1}{2D}(1 - (D + D')^{-1}) \left\{1 - \frac{D + 3D'}{2}\right\}^{-1} (x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D} \{1 + (D + D') + (D + D')^2 + ...\} \left\{1 + \frac{D + 3D'}{2} + \left(\frac{D + 3D'}{2}\right. \right. ... 4xy + 2y^2)$$

$$= \frac{1}{2D} \left\{1 + (D + D') + (D + D')^2 + \frac{D + 3D'}{2} + \left(\frac{D + 3D'}{2}\right)^2 + \frac{(D + D')(D + 3D')}{2} ... \right\} (x^2 - 4xy + 2y^2$$

Now

$$P.I. = \frac{1}{F(D,D')}(x^2 - 4xy + 2y^2) =$$

$$\frac{1}{2D}(1-(D+D')^{-1})\left\{1 - \frac{D+3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$=$$

$$\frac{1}{2D}\{1+(D+D')+(D+D')^2+...\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3I}{2}\right.\right.$$

$$4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\right.$$

$$\left.\frac{(D+D')(D+3D')}{2}...\right\}(x^2 - 4xy + 2y^2$$

$$= \frac{1}{2D}\left\{1 + (D + D') + (D + D')^2 + \frac{D + 3D'}{2} + \left(\frac{D + 3D'}{2}\right)^2 + \right.$$

$$\left. \frac{(D + D')(D + 3D')}{2} \ldots \right\} (x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2} \ldots \right\} (x^2 - 4xy$$

$$= \frac{1}{2D}\left\{(x^2 - 4xy + 2y^2) + 3(x - 2y) + 5(2y - 2x) + \frac{7}{2} + 19 - 2\right.$$

$$= \frac{1}{2D}\left\{1 + (D+D') + (D+D')^2 + \frac{D+3D'}{2} + \left(\frac{D+3D'}{2}\right)^2 + \frac{(D+D')(D+3D')}{2}\ldots\right\}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2}\ldots\right\}(x^2 - 4xy$$

$$= \frac{1}{2D}\left\{(x^2 - 4xy + 2y^2) + 3(x-2y) + 5(2y-2x) + \frac{7}{2} + 19 - 2$$

$$= \frac{1}{2D}\left\{1 + (D+D') + (D+D')^2 + \frac{D+3D'}{2} + \left(\frac{D+3D'}{2}\right)^2 + \frac{(D+D')(D+3D')}{2} \ldots\right\}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2} \ldots\right\}(x^2 - 4xy$$

$$= \frac{1}{2D}\left\{(x^2 - 4xy + 2y^2) + 3(x-2y) + 5(2y-2x) + \frac{7}{2} + 19 - 2\right.$$

$$= \frac{1}{2D}\left\{1 + (D+D') + (D+D')^2 + \frac{D+3D'}{2} + \left(\frac{D+3D'}{2}\right)^2 + \frac{(D+D')(D+3D')}{2} \ldots\right\}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left\{1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2} \ldots\right\}(x^2 - 4xy$$

$$= \frac{1}{2D}\left\{(x^2 - 4xy + 2y^2) + 3(x - 2y) + 5(2y - 2x) + \frac{7}{2} + 19 - 2\right.$$

$$= \frac{1}{2D}\left\{x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

Therefore the required solution is

$$z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) + \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

$$= \frac{1}{2D}\left\{x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

Therefore the required solution is

$$z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) + \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

$$= \frac{1}{2D}\left\{x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

Therefore the required solution is

$$z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) + \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

$$= \frac{1}{2D}\left\{x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

Therefore the required solution is

$$z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) + \frac{1}{2}\left\{\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4xy + \frac{x}{2}\right\}.$$

Exercise

Solve the following PDE:

(1) $(D+D'-1)(D+2D'-3)z = 4+3x+6y$ **Ans.** $z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + 6 + x + 2y$

(2) $(D^2 - D'^2 - 3D + 3D')z = xy$ **Ans.** $z = f_1(y+x) + e^{3x} f_2(y-x) - (1/6)x^2y - (x^2/9) - (2x/27) - (x^3/18)$.

(3) $r - t + p + 3q - 2 = x^2y$ **Ans.** $z = e^{-2x} f_1(y+x) + e^x f_2(y-x) - (4x^2y + 4xy + 6x^2 + 6y + 12x + 21)/8$.

Case-IV: When $\phi(x,y) = Ve^{ax+by}$, where V is a ny function of x and y.

Then

$$P.I. = \frac{1}{F(D,D')}Ve^{ax+by} = e^{ax+by}\frac{1}{F(D+a,D'+b)}V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Case-IV: When $\phi(x, y) = Ve^{ax+by}$, where V is a ny function of x and y.
Then

$$P.I. = \frac{1}{F(D, D')} Ve^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Case-IV: When $\phi(x,y) = Ve^{ax+by}$, where V is a ny function of x and y.
Then

$$P.I. = \frac{1}{F(D,D')}Ve^{ax+by} = e^{ax+by}\frac{1}{F(D+a,D'+b)}V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Case-IV: When $\phi(x, y) = Ve^{ax+by}$, where V is a ny function of x and y.

Then

$$P.I. = \frac{1}{F(D, D')} Ve^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Case-IV: When $\phi(x,y) = Ve^{ax+by}$, where V is a ny function of x and y.
Then

$$P.I. = \frac{1}{F(D,D')}Ve^{ax+by} = e^{ax+by}\frac{1}{F(D+a,D'+b)}V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Case-IV: When $\phi(x, y) = Ve^{ax+by}$, where V is a ny function of x and y.
Then

$$P.I. = \frac{1}{F(D, D')} Ve^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} V.$$

Example

Solve the PDE $(D^2 - D')z = xe^{ax+a^2y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

Non-Homogeneous Linear Partial Differential ...

Dr. G.K. Prajapati

LNJPIT, Chapra

Introduction

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2-k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

So that $D^2 z = \sum Ah^2 e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2 e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2 y}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

So that $D^2 z = \sum Ah^2 e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2 e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2 y}.$$

So that $D^2 z = \sum Ah^2 e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2 e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2 y}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$. By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$P.I. = \frac{1}{F(D, D')}\phi(x, y) = \frac{1}{(D^2 - D')} x e^{ax+a^2y} = e^{ax+a^2y} \frac{1}{((D+a)^2 - (D' + a^2))} x$$

$$= e^{ax+a^2y} \frac{1}{D^2 + 2aD - D'} x = e^{ax+a^2y} \frac{1}{2aD} \frac{1}{1 + \left(\frac{D^2 - D'}{2aD}\right)} x$$

$$= e^{ax+a^2y} \frac{1}{2aD} \left[1 + \left(\frac{D^2 - D'}{2aD}\right)\right]^{-1} x$$

$$P.I. = \frac{1}{F(D, D')}\phi(x, y) = \frac{1}{(D^2 - D')} x e^{ax+a^2y} = e^{ax+a^2y}\frac{1}{((D+a)^2 - (D'+a^2))}x$$

$$= e^{ax+a^2y}\frac{1}{D^2 + 2aD - D'}x = e^{ax+a^2y}\frac{1}{2aD}\frac{1}{1+\left(\frac{D^2-D'}{2aD}\right)}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1}x$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')} x e^{ax+a^2y} = e^{ax+a^2y}\frac{1}{((D+a)^2 - (D'+a^2))}x$$

$$= e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x = e^{ax+a^2y}\frac{1}{2aD}\frac{1}{1+\left(\frac{D^2-D'}{2aD}\right)}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1}x$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{((D+a)^2 - (D'+a^2))}x$$

$$= e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x = e^{ax+a^2y}\frac{1}{2aD}\frac{1}{1+\left(\frac{D^2-D'}{2aD}\right)}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1}x$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2-D')}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{((D+a)^2-(D'+a^2))}x$$

$$= e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x = e^{ax+a^2y}\frac{1}{2aD}\frac{1}{1+\left(\dfrac{D^2-D'}{2aD}\right)}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1}x$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{((D+a)^2 - (D'+a^2))}x$$

$$= e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x = e^{ax+a^2y}\frac{1}{2aD}\frac{1}{1+\left(\frac{D^2-D'}{2aD}\right)}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1+\left(\frac{D^2-D'}{2aD}\right)\right]^{-1}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-\dots\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+\dots\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z=\sum Ae^{hx+h^2y}+e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-...\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+...\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z = \sum Ae^{hx+h^2y} + e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-\ldots\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+\ldots\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z=\sum Ae^{hx+h^2y}+e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-\ldots\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+\ldots\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z = \sum Ae^{hx+h^2y} + e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-\ldots\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+\ldots\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z=\sum Ae^{hx+h^2y}+e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[1-\left(\frac{D^2-D'}{2aD}\right)+\left(\frac{D^2-D'}{2aD}\right)^2-\ldots\right]x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\left(\frac{D}{2a}x-\frac{D'}{2aD}x\right)+\ldots\right]$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left[x-\frac{1}{2a}\right] \implies = e^{ax+a^2y}\frac{1}{2a}\left[\frac{x^2}{2}-\frac{1}{2a}x\right]$$

Hence the required solution is

$$z = \sum Ae^{hx+h^2y} + e^{ax+a^2y}\left[\frac{x^2}{4a}-\frac{x}{4a^2}\right].$$

Example

Solve the PDE $(D - 3D' - 2)^2 z = 2e^{2x}\sin(y + 3x)$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$e^{2x}\left[f_1(y + 3x) + x f_2(y + 3x)\right]$, where f_1 and f_2 are arbitrary function.

Example

Solve the PDE $(D - 3D' - 2)^2 z = 2e^{2x}\sin(y + 3x)$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$e^{2x}[f_1(y + 3x) + xf_2(y + 3x)]$, where f_1 and f_2 are arbitrary function.

Example

Solve the PDE $(D - 3D' - 2)^2 z = 2e^{2x} \sin(y + 3x)$

The given equation is has reducible factor. Therefore, the complementary function (C.F.) is

$e^{2x}\left[f_1(y+3x) + x f_2(y+3x)\right]$, where f_1 and f_2 are arbitrary function.

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2} 2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2} 2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$
$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Now

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D-3D'-2)^2}2e^{2x}\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}2e^{2x+0y}\sin(y+3x) \implies$$

$$2e^{2x+0y}\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\sin(y+3x) \implies 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x)$$

Hence the required solution is

$$z = e^{2x}\left[f_1(y+3x) + xf_2(y+3x)\right] + 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x).$$

Exercise

Solve the following PDE:

(1) $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$ **Ans.** $z = \sum Ae^{hx+ky} + (4/3)e^{x+y}\sin(x+y)$, where h and k are related by $3h^2 - 2k^2 + h - 1$.

(2) $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y+3x)$ **Ans.** $z = e^{2x}f_1(y+3x) + xf_2(y+3x) + x^2e^{2x}\tan(y+3x)$.

(3) $r - 3s + 2t - p + 2q = (2+4x)e^{-y}$ **Ans.** $z = f_1(y+2x) + e^x f_2(y+x) + x^2e^{-y}$.

Case-V: When $\phi(x, y) = e^{ax+by}$ and $F(a, b) = 0$.

Then

$$P.I. = \frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(D, D')}e^{ax+by}.1 = e^{ax+by}\frac{1}{F(D+a, D'+b)}.x^0y^0$$

Case-V: When $\phi(x, y) = e^{ax+by}$ and $F(a, b) = 0$.
Then

$$P.I. = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(D, D')} e^{ax+by}.1 = e^{ax+by} \frac{1}{F(D+a, D'+b)}.x^0 y^0$$

Case-V: When $\phi(x,y) = e^{ax+by}$ and $F(a,b) = 0$.
Then

$$P.I. = \frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(D,D')}e^{ax+by}.1 = e^{ax+by}\frac{1}{F(D+a,D'+b)}.x^0y^0$$

Case-V: When $\phi(x,y) = e^{ax+by}$ and $F(a,b) = 0$.
Then

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(D,D')} e^{ax+by}.1 = e^{ax+by} \frac{1}{F(D+a, D'+b)}.x^0 y^0$$

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2 z = \sum Ah^2 e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2 z = \sum Ah^2 e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

Example

Solve the PDE $(D^2 - D')z = e^{x+y}$.

Solution: The given equation can not be written as linear factors. Hence it's complementary function (C.F.) is taken as a trial solution

$$z = \sum Ae^{hx+ky}.$$

So that $D^2z = \sum Ah^2e^{hx+ky}$ and $D'z = \sum Ake^{hx+ky}$.By Putting these values in given equation $(D^2 - D')z = 0$, we have

$$\sum Ah^2 e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2-k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2-k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2-k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$\sum Ah^2e^{hx+ky} - \sum Ake^{hx+ky} = 0 \implies \sum A(h^2 - k)e^{hx+ky} = 0.$$

$$h^2 - k = 0 \implies k = h^2.$$

Hence

$$C.F. = \sum Ae^{hx+h^2y}.$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1))}.1$$

$$= e^{x+y}\frac{1}{D^2 + 2D - D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1 + \left[\left(\frac{D^2 - D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1 + \left(\frac{D^2 - D'}{2D}\right)\right]^{-1}.1$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1))}.1$$

$$= e^{x+y}\frac{1}{D^2+2D-D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1+\left[\left(\frac{D^2-D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1+\left(\frac{D^2-D'}{2D}\right)\right]^{-1}.1$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1)}.1$$

$$= e^{x+y}\frac{1}{D^2 + 2D - D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1 + \left[\left(\frac{D^2 - D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1 + \left(\frac{D^2 - D'}{2D}\right)\right]^{-1}.1$$

$$P.I. = \frac{1}{F(D, D')}\phi(x, y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1)}.1$$

$$= e^{x+y}\frac{1}{D^2 + 2D - D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1 + \left[\left(\frac{D^2 - D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1 + \left(\frac{D^2 - D'}{2D}\right)\right]^{-1}.1$$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1)}.1$$

$$= e^{x+y}\frac{1}{D^2 + 2D - D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1 + \left[\left(\frac{D^2 - D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1 + \left(\frac{D^2 - D'}{2D}\right)\right]^{-1}.1$$

$$P.I. = \frac{1}{F(D, D')}\phi(x, y) = \frac{1}{(D^2 - D')}e^{x+y}.1 = e^{x+y}\frac{1}{((D+1)^2 - (D'+1)}.1$$

$$= e^{x+y}\frac{1}{D^2 + 2D - D'}.1 = e^{x+y}\frac{1}{2D}\frac{1}{1 + \left[\left(\frac{D^2 - D'}{2D}\right)\right]}.1$$

$$= e^{x+y}\frac{1}{2D}\left[1 + \left(\frac{D^2 - D'}{2D}\right)\right]^{-1}.1$$

Non-Homogeneous Linear Partial Differential ...

Dr. G.K. Prajapati

LNJPIT, Chapra

Introduction

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D^2-D'}{2D}\right)+\left(\frac{D^2-D'}{2D}\right)^2-...\right].1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D}{2}.1-\frac{D'}{2D}.1\right)+...\right]$$

$$= e^{x+y}\frac{1}{2D}(1) \implies = e^{x+y}\left(\frac{1}{2}\right)x$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D^2-D'}{2D}\right)+\left(\frac{D^2-D'}{2D}\right)^2-...\right].1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D}{2}.1-\frac{D'}{2D}.1\right)+...\right]$$

$$= e^{x+y}\frac{1}{2D}(1) \implies = e^{x+y}\left(\frac{1}{2}\right)x$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D^2-D'}{2D}\right)+\left(\frac{D^2-D'}{2D}\right)^2-...\right].1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D}{2}.1-\frac{D'}{2D}.1\right)+...\right]$$

$$= e^{x+y}\frac{1}{2D}(1) \implies = e^{x+y}\left(\frac{1}{2}\right)x$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D^2-D'}{2D}\right)+\left(\frac{D^2-D'}{2D}\right)^2-...\right].1$$

$$= e^{x+y}\frac{1}{2D}\left[1-\left(\frac{D}{2}.1-\frac{D'}{2D}.1\right)+...\right]$$

$$= e^{x+y}\frac{1}{2D}(1) \implies = e^{x+y}\left(\frac{1}{2}\right)x$$

Hence the required solution is

$$z = \sum Ae^{hx+h^2y} + \frac{x}{2}e^{x+y}.$$

Hence the required solution is

$$z = \sum A e^{hx+h^2y} + \frac{x}{2} e^{x+y}.$$

Exercise

Solve the following PDE:

(1) $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$ **Ans.** $z = f_1(y+x) + e^{3x} f_2(y-x) - xe^{x+2y}$.

(2) $(D^2 - D')z = e^{2x+y}$ **Ans.** $z = \sum Ae^{hx+h^2y} - \frac{1}{3}e^{2x+y}$

(3) $r - 4s + 4t + p - 2q = e^{x+y}$ **Ans.** $z = f_1(y+2x) + e^{-x} f_2(y+2x) - xe^{x+y}$.

Thanks !!!