WEEK 3: Lecture Notes

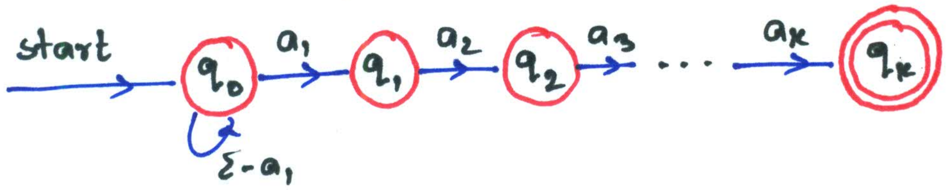
ε - NFA

- Allows a transition on ε, the empty string (spontaneous transition without receiving an input)
- Doesnot expand the class of languages that can be accepted by FA.
- It gives us some added "programing convenience"
- Closely related to regular expressions and useful in proving the equivalence between classes of languages accepted by FA and by r.e.

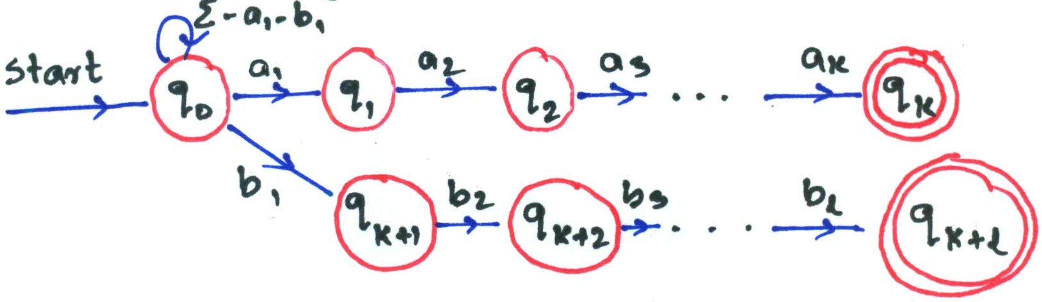
An application: Text Search

- Σ : every printable ASCII characters
- $a_1, a_2, \cdots a_k$: a Keyword , $a_i \in \Sigma$

- NFA that recognizes $a_1 a_2 \ldots a_k$



- NFA that recognizes $a_1 a_2 \cdots a_k$ or $b_1 b_2 \cdots b_l$



- DFA that recognizes $a_1 a_2 \ldots a_k$ (use subset construction)

1. NFA start state q_0

   DFA start state $\{q_0\}$.

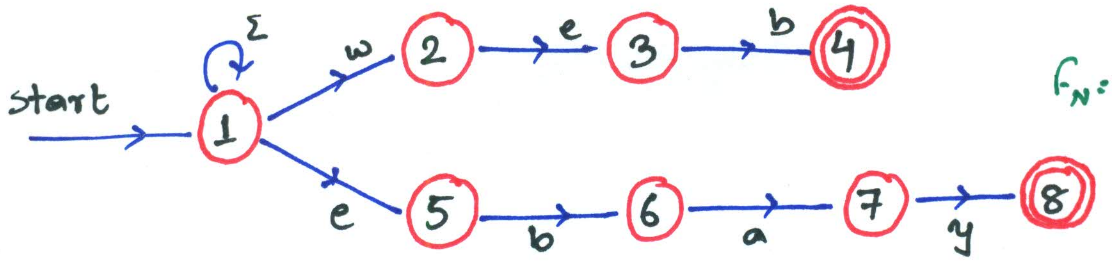
2. if $q_0 \xrightarrow{a_1 a_2 \cdots a_m} p$ in NFA then

   the DFA state $\{q_0, p, r_1, r_2 \ldots r_t\}$

   where $q_0 \xrightarrow[\text{of } a_1 a_2 \ldots a_m]{\text{a suffix}} r_i$, $i = 1, 2, \ldots, t$

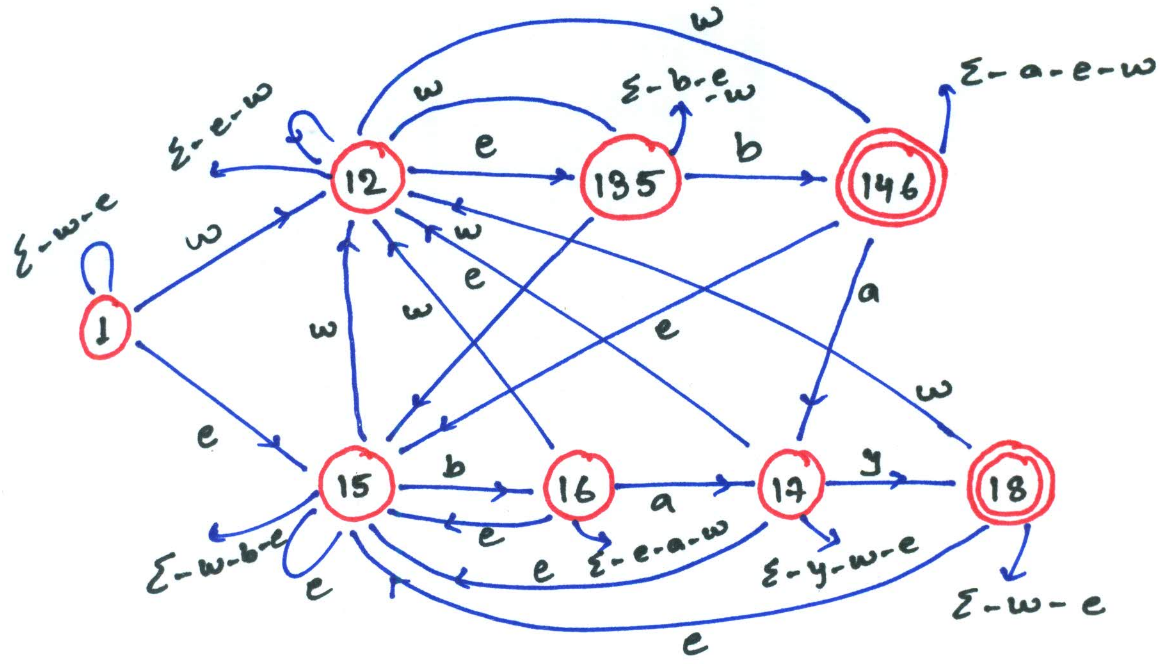
i.e. $r_1 r_2 \ldots r_t \rightarrow$ every other state of the NFA that is reachable from q_0 following a path whose labels are a suffix of $a_1 a_2 \ldots a_m$

Example web ebay



$F_N = \{4, 8\}$

DFA state $\{a, b, c\} \rightarrow abc$

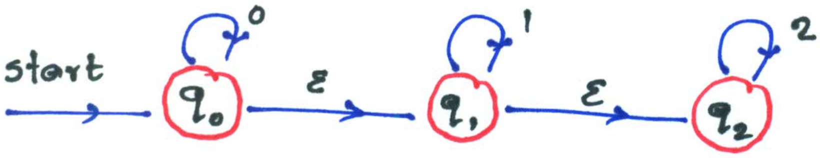


- # of states of the NFA = # of states of DFA
- Transitions are more in DFA

Finite Automata with ε-moves (ε-NFA)

- quintuple $(Q, \Sigma, \delta, q_0, F)$ — same as NFA, only differs in δ
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^Q$

Example:



ε- NFA for $0^* 1^* 2^*$

accepting the language

$L = \{ w \mid w$ consits of any no. of 0's followed by any no. of 1's followed by any no. of 2's $\}$

- 002 is accepted by the ε-NFA by the path

$q_0, q_0, q_0, q_1, q_2, q_2$ with arcs labeled $0, 0, \varepsilon, \varepsilon, 2$

$$\underset{\text{start}}{q_0} \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{\varepsilon} q_1 \xrightarrow{\varepsilon} q_2 \xrightarrow{2} q_2$$

	inputs			
States	0	1	2	ε
q_0	$\{q_0\}$	ϕ	ϕ	$\{q_1\}$
q_1	ϕ	$\{q_1\}$	ϕ	$\{q_2\}$
q_2	ϕ	ϕ	$\{q_2\}$	ϕ

ε - closure

ε - closure $(q) = \{ p \mid$ there is a path from q to p with all arcs labeled $\varepsilon \}$

state

Example:

ε - closure $(q_0) = \{ q_0, q_1, q_2 \}$ | for the previous example.

$q_0 \in \varepsilon$ - closure (q_0) as path consisting of q_0 along has no arc, hence trivially all it's arcs are labeled ε.

Recursively

1. $q \in \varepsilon$ - closure (q)
2. if $p \in \varepsilon$ - closure (q) and $r \in \delta(p, \varepsilon)$ then $r \in \varepsilon$ - closure (q)
3. if $p \in \varepsilon$ - closure (q), then $\delta(p, \varepsilon) \subseteq \varepsilon$ - closure (q)

$\delta \rightarrow$ transition function of the ε-NFA involved

ε - closure (P)

$P \rightarrow$ a set of states

$$\bigcup_{p \in P} \varepsilon\text{- closure } (p)$$

Extended transition function for ε-NFA ($\hat{\delta}$)

1. $\hat{\delta}(q, \varepsilon) = \varepsilon\text{-closure}(q)$

2. $\hat{\delta}(q, xa) = \varepsilon\text{-closure}(P)$, where

   $x \in \Sigma^*, a \in \Sigma$

   $P = \{p \mid \text{for some } r \in \hat{\delta}(q,x) \ p \text{ is in } \delta(r,a)\}$

   $= \{p \mid \text{i.e. } p \in \delta(\hat{\delta}(q,x), a)\}$

   $= \delta(\hat{\delta}(q,x), a)$

<u>Extend δ and $\hat{\delta}$ to sets of states.</u>

3. $\delta(R, a) = \bigcup_{q \in R} \delta(q, a)$

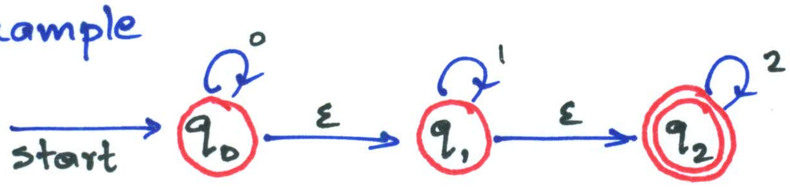
4. $\hat{\delta}(R, w) = \bigcup_{q \in R} \hat{\delta}(q, w)$

   for sets of states $R \subset Q$.

- $\hat{\delta}(q, a)$ not necessarily equal to $\delta(q,a)$ for ε-NFA as $\hat{\delta}(q,a) \rightarrow$ all states reachable from q by paths labeled a (including paths with arcs labeled ε)

  $\delta(q, a) \rightarrow$ only states reachable from q by arcs labeled a

Example



$$\hat{\delta}(q_0, 0) = \hat{\delta}(q_0, \varepsilon 0)$$

$$= \varepsilon\text{-closure}\left(\delta(\hat{\delta}(q_0, \varepsilon), 0)\right)$$

$$= \varepsilon\text{-closure}\left(\delta(\varepsilon\text{-closure}(q_0), \varepsilon)\right)$$

$$= \varepsilon\text{-closure}\left(\delta(\{q_0, q_1, q_2\}), 0)\right)$$

$$= \varepsilon\text{-closure}\left(\{q_0\} \cup \phi \cup \phi\right)$$

$$= \varepsilon\text{-closure}\left(\{q_0\}\right)$$

$$= \{q_0, q_1, q_2\}$$

$$\hat{\delta}(q_0, 01) = \varepsilon\text{-closure}\left(\delta(\hat{\delta}(q_0, 0), 1)\right)$$

$$= \varepsilon\text{-closure}\left(\delta(\{q_0, q_1, q_2\}, 1)\right)$$

$$= \varepsilon\text{-closure}\left(\phi \cup \{q_1\}, \cup \phi\right)$$

$$= \varepsilon\text{-closure}\left(\{q_1\}\right)$$

$$= \{q_1, q_2\}$$

Similary:

$$\hat{\delta}(q_0, 2) = \{q_2\}.$$

ε- NFA

- allows a transition spontaneously without receiving an input symbol
- this new capability gives some added programming convenience; without expanding the class of languages that can be accepted by finite automata.
- useful in proving the equivalence between the classes of languages accepted by finite automata and by regular expressions.

Equivalence of NFA's and DFA's

Theorem:

L is accepted by some ε-NFA iff L is accepted by some NFA

Proof:

(if) $M = (Q, \Sigma, \delta, q_0, F)$, an NFA

can be interpreted as an ε-NFA

$$M' = (Q, \Sigma \cup \{\varepsilon\}, \delta', q_0, F)$$

where δ' is defined by

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } a \neq \varepsilon,\ a \in \Sigma \\ \phi & \text{otherwise} \end{cases}$$

(only if)

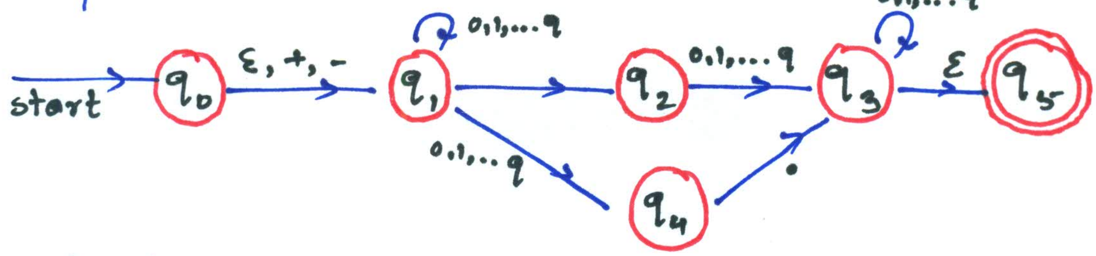
if $M = (Q, \Sigma \cup \{\varepsilon\}, \delta, q_0, F)$ is ε-NFA

an NFA can be constructed

$M' = (Q, \Sigma, \delta', q_0, F')$ where

- $\delta'(q, a) = \hat{\delta}(q, a)$ for $q \in Q$, $a \in \Sigma$
- $F' = \begin{cases} F \cup \{q_0\} & \text{if } \varepsilon\text{-closure}(q_0) \cap F \neq \phi \\ F & \text{otherwise.} \end{cases}$

Example (ε- NFA to NFA construction)



$$F' = \begin{cases} F \cup \{q_0\}, & \text{if } \varepsilon\text{-}c(q_0) \cap F \neq \phi \\ F, & \text{otherwise} \end{cases}$$

NFA

δ'	+	−	•	0 − 9
→ q_0	$\{q_1\}$	$\{q_1\}$	$\{q_2\}$	$\{q_1\}$
q_1	ϕ	ϕ	$\{q_2\}$	$\{q_1, q_4\}$
q_2	ϕ	ϕ	ϕ	$\{q_3, q_5\}$
q_3	ϕ	ϕ	ϕ	$\{q_3, q_5\}$
q_4	ϕ	ϕ	$\{q_3, q_5\}$	ϕ
q_5	ϕ	ϕ	ϕ	ϕ

$$\delta'(q_0, +) = \hat{\delta}(q_0, +) = \hat{\delta}(q_0, \varepsilon +) = \varepsilon\text{-}c(\delta(\hat{\delta}(q_0, \varepsilon), +))$$
$$= \varepsilon\text{-}c(\delta(\varepsilon\text{-}c(q_0), +))$$

$\varepsilon c(q_0) = \{q_0, q_1\}$ $\xrightarrow{+}$ $\{q_1\} \cup \phi \xrightarrow{\varepsilon.c.} \{q_1\}$

$\varepsilon c(q_1) = \{q_1\}$ $\xrightarrow{-}$ $\{q_1\} \cup \phi \xrightarrow{\varepsilon.c.} \{q_1\}$

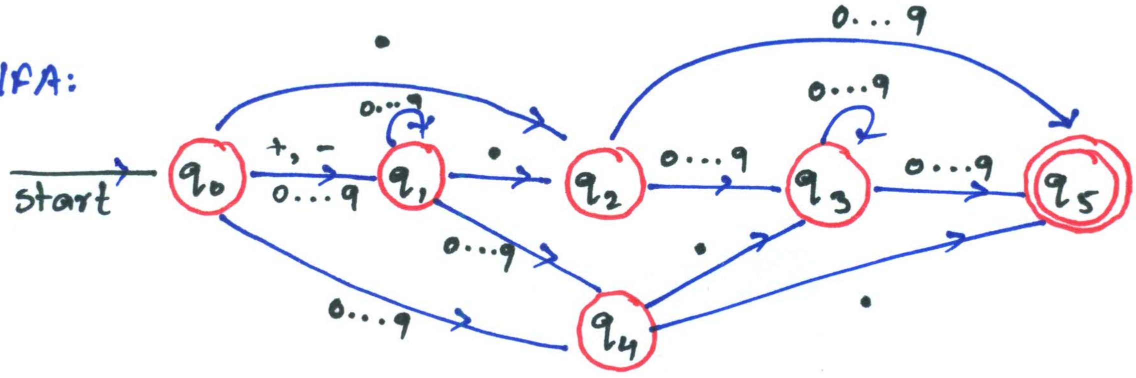
$\varepsilon c(q_2) = \{q_2\}$ $\xrightarrow{\bullet}$ $\phi \cup \{q_2\} \xrightarrow{\varepsilon.c.} \{q_2\}$

$\varepsilon c(q_3) = \{q_3, q_5\}$ $\xrightarrow{0..9}$ $\phi \cup \{q_1, q_4\} \xrightarrow{\varepsilon.c.} \{q_1, q_4\}$.

$\varepsilon c(q_4) = \{q_4\}$

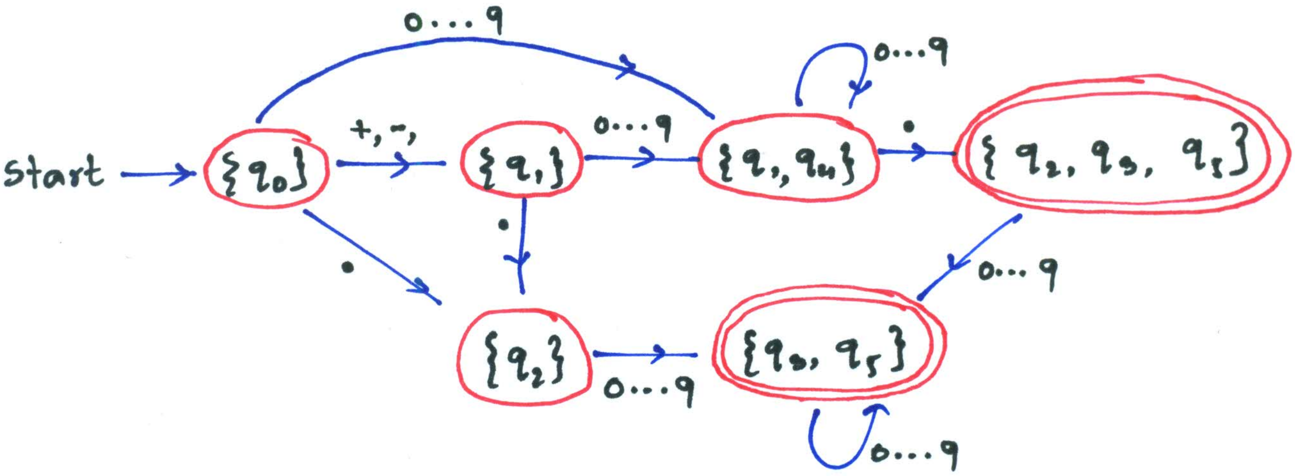
$\varepsilon c(q_5) = \{q_5\}$

NFA:



NFA to DFA (subset construction)

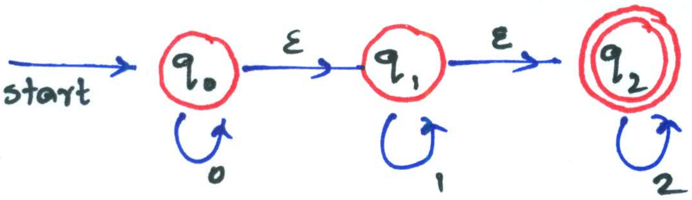
δ	+	−	.	0...9
→ $\{q_0\}$	$\{q_1\}$	$\{q_1\}$	$\{q_2\}$	$\{q_1, q_4\}$
$\{q_1\}$	ϕ	ϕ	$\{q_2\}$	$\{q_1, q_4\}$
$\{q_2\}$	ϕ	ϕ	ϕ	$\{q_3, q_5\}$
$\{q_1, q_4\}$	ϕ	ϕ	$\{q_1, q_3, q_5\}$	$\{q_1, q_4\}$
* $\{q_3, q_5\}$	ϕ	ϕ	ϕ	$\{q_3, q_5\}$
* $\{q_1, q_3, q_5\}$	ϕ	ϕ	ϕ	$\{q_3, q_5\}$



(dead states and all transition to the dead state is not shown)

Example (ε-NFA to NFA conversion)

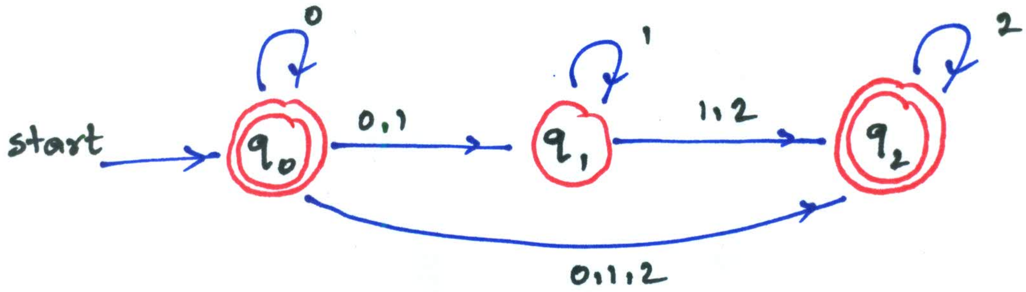
ε-NFA $(Q, \Sigma \cup \{\varepsilon\}, \delta, q_0, F)$



δ	0	1	2	ε
→ q_0	$\{q_0\}$	ϕ	ϕ	$\{q_1\}$
q_1	ϕ	$\{q_1\}$	ϕ	$\{q_2\}$
* q_2	ϕ	ϕ	$\{q_2\}$	ϕ

Equivalent NFA $(Q, \Sigma, \delta', q_0, F')$

δ'	0	1	2
q_0	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
q_1	ϕ	$\{q_1, q_2\}$	$\{q_2\}$
q_2	ϕ	ϕ	$\{q_2\}$



Claim I:

$L(M) = L(M')$

where M is ε-NFA $(Q, \Sigma \cup \{\varepsilon\}, \delta, q_0, F)$

and M' is NFA $(Q, \Sigma, \delta', q_0, F')$

where, $\delta'(q, a) = \hat{\delta}(q, a)$

and $F' = \begin{cases} F \cup q_0 & \text{if } \varepsilon\text{-closure}(q_0) \cap F \neq \phi \\ F & \text{otherwise} \end{cases}$

- M accepts a string w iff $\hat{\delta}(q_0, w) \cap F \neq \phi$
- M' accepts a string w iff $\hat{\delta}'(q_0, w) \cap F \neq \phi$

$\Rightarrow$ $L(M) = L(M')$ iff $\hat{\delta}(q_0, w) = \hat{\delta}'(q_0, w)$

Claim I holds if Claim II holds

Claim II: $\underset{\text{NFA}}{\hat{\delta}'(q_0, w)} = \underset{\varepsilon\text{-NFA}}{\hat{\delta}(q_0, w)}$ for a string w — (1)

We prove this by induction on $|w|$

Base: $|w| = 1$ i.e. $w = a \in \Sigma$

$$\hat{\delta}'(q_0, w) = \hat{\delta}'(q_0, a)$$

$$= \delta'(q_0, a) = \hat{\delta}(q_0, a)$$ by our construction

$$= \hat{\delta}(q_0, w)$$

Induction:

$|w| > 1$, $w = xa$, $x \in \Sigma^*$, $a \in \Sigma$

$$\hat{\delta}'(q_0, x) = \underbrace{\hat{\delta}(q_0, x)}_{(2)} \quad \text{by induction hypothesis}$$

Now, LHS of 1:

$$\hat{\delta}(q_0, w) = \hat{\delta}'(q_0, xa) = \delta'(\hat{\delta}'(q_0, x), a)$$

$$= \delta'(\hat{\delta}(q_0, x), a) \quad \text{by (2)}$$

where $p = \hat{\delta}(q_0, x)$

$$= \delta'(P, a) = \bigcup_{r \in P} \delta'(r, a)$$

$$= \bigcup_{r \in P} \hat{\delta}(r, a)$$

by construction

R.H.S. of 1.

$$\hat{\delta}(q_0, w) = \hat{\delta}(q_0, xa)$$

$$= \varepsilon\text{-closure}\{p \mid p \in \delta(r, a) \text{ for some } r \in \hat{\delta}(q_0, x)\}$$

$$\bigcup_{p \in \delta(\hat{\delta}(q_0, x), a)} \varepsilon\text{-closure}(p)$$

$$= \bigcup_{p \in \delta(p, a)} \varepsilon\text{-closure}(p)$$

$$= \bigcup_{p \in \delta(r, a) \text{ for some } r \in p} \varepsilon\text{-closure}(p)$$

= L.H.S.

Equivalence of ε-NFA's and DFA's

Theorem:

A language L is acceptable by some ε-NFA iff L is accepted by some DFA.

Proof:

Given a DFA $D = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(D)$

we can construct ε-NFA

$$E = (Q, \Sigma \cup \{\varepsilon\}, \delta', q_0, F)$$

where δ' is defined by

$$\delta'(q,a) = \begin{cases} \{\delta(q,a)\} & \text{if } a \neq \varepsilon,\ a \in \Sigma \\ \phi & \text{if } a = \varepsilon,\ a \notin \Sigma \end{cases}$$

E explicitly states that there are transitions out of any state on ε $\Rightarrow$ $L = L(D) = L(E)$.

Now, given ε-NFA $E = (Q_E, \Sigma \cup \{\varepsilon\}, \delta_E, q_0, F_E)$

we can construct DFA $D = (Q_D, \Sigma, \delta_D, q_D, F_D)$

as follows:

1. $Q_D = \{ S \subseteq Q_E \mid S = \varepsilon\text{-closure}(S) \}$: ε-closed subsets of Q

2. $q_D = \varepsilon\text{-closure}(q_0)$: rule differs from subset construction

3. $F_D = \{ S \in Q_D \mid S \cap F_E \neq \phi \}$

4. $\delta_D(s, a)$, $s \in Q_D$, $a \in \Sigma$, defined by:

(i) let $s = \{p_1, p_2, \dots, p_k\}$, note that $s =$ ε-closure(s)

(ii) compute $\bigcup_{i=1}^{k} \delta_E(p_i, a)$, let this set be $\{r_1, r_2, \dots r_m\}$

(iii) $\delta_D(s, a) = \bigcup_{j=1}^{m} \varepsilon\text{-closure}(r_j)$

$$\delta_D(s,a) = \bigcup_{s \in S} \hat{\delta}_E(s, a)$$

$$= \bigcup_{s \in S} \varepsilon\text{-closure}\left(\delta_E(\hat{\delta}_E(s,\varepsilon), a)\right)$$

$$= \bigcup_{s \in S} \varepsilon\text{-closure}\left(\delta_E(\varepsilon\text{-closure}(s), a)\right)$$

$$= \bigcup_{i=1}^{k} \varepsilon\text{-closure}\left(\delta_E(\varepsilon\text{-closure}(p_i), a)\right)$$

$$= \bigcup_{i=1}^{k} \varepsilon\text{-closure}\left(\delta_E(p_i, a)\right)$$

as $p_i \in S$ and $S =$ ε-closure
ε-closure$(p_i) \subseteq S$

$$= \varepsilon\text{-closure}\left(\bigcup_{i=1}^{k} \delta_E(p_i, a)\right)$$

$$= \varepsilon\text{-closure}(r_1, r_2, \dots, r_m)$$

$$= \bigcup_{j=1}^{m} \varepsilon\text{-closure}(r_j)$$