Week 1: Lecture Notes

Finite State Systems

- State : Summarizes the information concerning past inputs that is needed to determine the behaviour of the systems on subsequent inputs.

Example:

- Elevator (Control Mechanism)
  - current floor
  - up / down
  - collection not yet satisfied requests for service

  states

- Switching circuit, such as the control unit of a computer
  - finite number of gates each of which is either on/off
  - n gates $\Rightarrow$ 2^n assignments of 0 or 1 to various gates.

- Programs such as text editors and the lexical analyzers used in most compilers are often designed as finite state system.

- a lexical analyzer scans the symbol of a computer program to locate strings of characters corresponding to identifiers, numerical constants, reserved words and so on.

- The lexical analyzer needs to remember only a finite amount of information, such as how long a prefix of a reserved word it has seen since startup.

• Computer itself can be viewed as a finite state system.

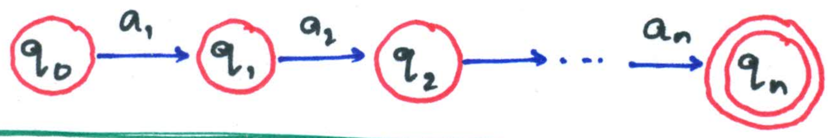
• Human brain → finite state system.
  # of brain cells or neurons is limited. (2^{35} at most)

Finite Automata / Automaton (FA)

- The most basic model of a computer
- Computer without memory / the amount of memory is fixed, regardless of the size of the input.
- involves
  - → states
    - → accept states
    - → reject states
  - → transition among states in response to inputs

- an abstract computing device that recognizes a collection of strings.

  $w = a_1 a_2 \cdots a_n$ → string



if accept state (i.e. $q_n \in F$) w is recognized by the FA
otherwise FA rejects w

A finite automata modeling recognition of the string $w = a_1 a_2 \cdots a_n$ may be a part of lexical analyzer.

Applications of FA

- useful model for many important kinds of hardware and software.
  - design of lexical analyzer of a typical compiler.

- software for designing and checking the behaviour of digital circuits/protocols.

- software for scanning large bodies of text (e.g. web page) to find occurrences of words, phrases or other patterns (pattern recognition)

- protocols (with finite number of states)
  - → communication protocol
  - → protocol for secure exchange of information

- lexical analyzer of a compiler:
  the compiler component that breaks the input text into logical units, e.g. identifiers, keywords and punctuations.

- Automata are essential to study the limits of computation:
  - → Decidable problems (solvable by computer) (What can a computer do at all?)
  - → Tractable problems (solvable by computer efficiently)
    (What can a computer do efficiently?) → time complexity is a slowly growing function in the size of input

    ↓ studies

    Intractable / tractable problems

Turing Machines: automata that models the power of real computers.

- allows us to study decidability → the question of what can or cannot be done by a computer
- also allows us to distinguish tractable (solvable in polynomial time) from intractable (not solvable in polynomial time) problems.

Content-free grammars and Push down automata

- useful tool for describing structure of programming languages and design of parser → another key portion of a compiler which deals with recursively nested featurs of the typical programming language (e.g. arithmetic, conditional etc.)

Regular Expressions

- useful for describing same patterns that can be represented by finite automata and design of lexical analyzer (compiler component that groups character into tokens)

Example: Unix-style regular expression

[A-Z][a-z] * [ ] [A-Z][A-Z]

Kolkata WB

[A-Z][a-z] *([ ] [A-Z] [a-z] *) *[ ][A-Z][A-Z]

Kolkata West Bengal IN

Deterministic Finite Automata (DFA)

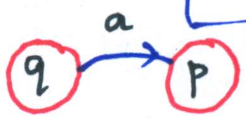
A 5-tuple $(Q, \Sigma, \delta, q_0, F)$

Q : a finite set of states

Σ : a finite input alphabet

δ : $Q \times \Sigma \to Q$, the transition function:

$p = \delta(q, a)$

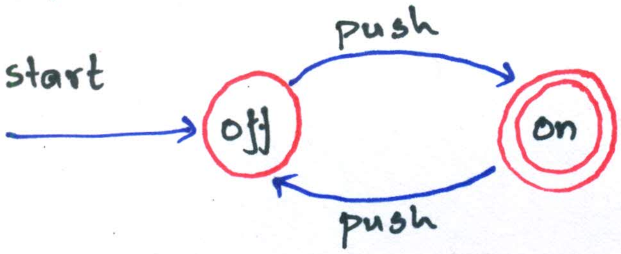


$q_0 \in Q$: the initial state

$F \subseteq Q$: the set of final/accepting states

- $\Sigma = \{0, 1\}$ binary
- $\Sigma = \{a, b, \ldots, z\}$ → lower case letters
- Σ = set of all ASCII characters.

Example: (A finite automata modeling an on/off switch)



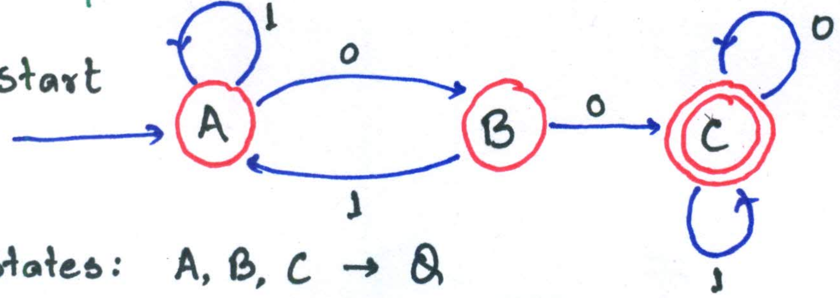
(Transition diagram)

- states: on, off → Q
- input : push → Σ
- accepting state: on → F
- initial state : off → q_0

δ	push
→ off	on
* on	off

Transition table

Example



States: A, B, C → Q
input: 0, 1 → Σ
accepting state: C → F
initial state : A → q_0

Transition table

δ	0	1
→ A	B	A
B	C	A
* C	C	C

- Alphabet (Σ)
  - a finite non-empty set of symbols

  e.g. $\Sigma = \{0,1\}$ → binary alphabet

- Strings / words (w)
  - a finite sequence of symbols

  e.g. 01101 is a string from the binary alphabet $\Sigma = \{0,1\}$

- Empty string (ε)
  - zero occurrences of symbols

- Length of a string w
  - $|w|$: # of positions of symbols in w

  e.g. $|\varepsilon| = 0$, $|01101| = 5$

- Convention:
  - lower case letters at the begining of the alphabet (or digits) → <u>symbols</u> e.g. $a, b, c, ..$
  - lower case letters near the end of the alphabet → <u>strings</u> e.g. w, x, y, z

- Concatenation of strings

  $x = a_1 a_2 \dots a_i$, $y = b_1 b_2 \dots b_j$

  $xy = a_1 a_2 \dots a_i b_1 b_2 \dots b_j$ → string of length $i+j$

  For any string w, we have

  $$w\varepsilon = \varepsilon w = w$$

Power of an alphabet

- Σ - an alphabet
- Σ^k - set of strings of length k, each of whose symbols is in Σ.

Example:

- $\Sigma = \{a, b, c\}$ - alphabet
- $\Sigma^0 = \varepsilon$ - ε is the only string of length 0
- $\Sigma^1 = \{a, b, c\}$ - strings of length 1.
- $\Sigma^2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$
- $\Sigma^3 =$ strings of length 3

- Σ^* - set of all strings over an alphabet Σ

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$$
$$= \{\varepsilon\} \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$$

Σ^+ - $\Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots$

So,

$$\Sigma^* = \{\varepsilon\} \cup \Sigma^+$$

Languages

- $L \subseteq \Sigma^*$ → a language over Σ

Example:

1. Languages of all strings consisting of n 0's followed by n 1's, for some $n \geq 0$ is

   $\{\varepsilon, 01, 0011, 000111\}$

   → $\{0^n 1^n \mid n \geq 0\}$

2. The set of strings of 0's and 1's with equal number of each $\{\varepsilon, 01, 10, 0011, 0101, 1001, \ldots\}$

3. The set of binary numbers whose value is a prime
   - $\{w \mid w$ is a binary integer that is prime$\}$
   - $\{10, 11, 101, 111, \ldots\}$

4. Σ^* - a language over Σ

5. ϕ - the empty language (a language over any alphabet)

6. $\{\varepsilon\}$ - a language over any alphabet.

   $\phi \neq \{\varepsilon\}$

   no string — are string of length 0

Language: may contain an infinite number of strings, but strings are drawn from one fixed, finite alphabet.

Problem - Decisional Problems

- Membership in a language
- Σ - an alphabet
- L - language over Σ

Problem: $L \rightarrow$ Given a string w in Σ^*, decide whether or not $w \in L$

Example:

Primality Testing - Given an integer decide whether it is prime or not

Reformulation: Express the problem by the language L_p consisting of all binary strings whose value as a binary number is prime.

$\Rightarrow$ Given a string w of 0's and 1's

output $\rightarrow$ YES if $w \in L_p$

$\rightarrow$ NO if $w \notin L_p$

How a DFA processes strings?

Consider a DFA $A = (Q, \Sigma, \delta, q_0, F)$ and a string $w = a_1 a_2 \dots a_n$

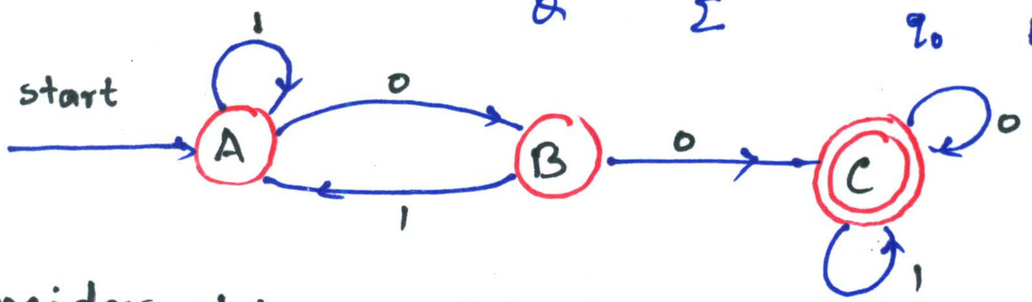
$$q_1 = \delta(q_0, a_1)$$
$$q_2 = \delta(q_1, a_2)$$
$$\vdots$$
$$q_i = \delta(q_{i-1}, a_i)$$
$$\vdots$$
$$q_n = \delta(q_{n-1}, a_n)$$

DFA A accepts the string $w = a_1 a_2 \dots a_n$ if $q_n \in F$

if not, A rejects w.

Example:

Consider DFA : $(\underbrace{\{A, B, C\}}_{Q}, \underbrace{\{0,1\}}_{\Sigma}, \delta, \underbrace{\{A\}}_{q_0}, \underbrace{\{C\}}_{F})$



Consider string $w = 101001$

current state	symbol road	New state
A	1	A
A	0	B
B	1	A
A	0	B
B	0	C
C	1	C

final state whichis the accept state

The above DFA accepts string w

Also, the above DFA

- doesnot accept 11101
- accepts 0001
- accepts all strings of 0's and 1's with two consecutive zeros somewhere.

- Language accepted by the given DFA
  ↳ regular language.

Extending the transition function to strings.

- DFA $\rightarrow (Q, \Sigma, \delta, q_0, F)$
- $\delta : Q \times \Sigma \rightarrow Q$ (transition function)
- $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$ (extended transition function)

we define it by induction on the length of input string.

Base : $\hat{\delta}(q, \varepsilon) = q$, we are in state q and read no input so we are still in state q

induction:

let $w \in \Sigma^*$, $w = xa$, $x \in \Sigma^*$, $a \in \Sigma$

(x: string consisting of all but the last symbol of w

a: last symbol of w)

Then

$$\hat{\delta}(q, w) = \hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

Example:

$w = 1101$

$= xa$; $x = 110$, $a = 1$

$$\hat{\delta}(q, w) = \delta(\hat{\delta}(q, x), a)$$

i.e. to compute $\hat{\delta}(q, w)$, first compute $\hat{\delta}(q, x)$

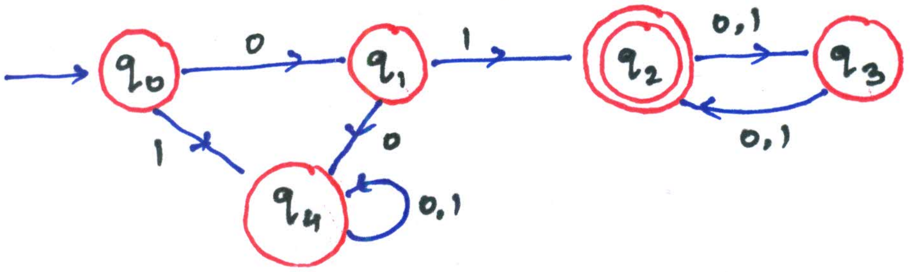
Example

Consider the DFA: $(Q = \{q_0, q_1, q_2, q_3, q_4\}, \Sigma = \{0,1\}$
$\delta, q_0, F = \{q_2\})$

the transition table:

δ	0	1
$\rightarrow q_0$	q_1	q_4
q_1	q_4	q_2
$* q_2$	q_3	q_3
q_3	q_2	q_2
q_4	q_4	q_4

So the transition diagram will be:



If $w = 011101$, then is w accepted by the above DFA, i.e. is $\hat{\delta}(q_0, w) \in F$?

$\Rightarrow$ chech each prefix x of $w = 011101$ starting at 1 and going in increasing size.

$\hat{\delta}(q_0, \varepsilon) = q_0$

$\hat{\delta}(q_0, 0) = \delta(\hat{\delta}(q_0, \varepsilon), 0)$
$= \delta(q_0, 0) = q_1$

$\hat{\delta}(q_0, 01) = \delta(\hat{\delta}(q_0, 0), 1)$
$= \delta(q_1, 1)$
$= q_2$

$\hat{\delta}(q_0, 011) = \delta(\hat{\delta}(q_0, 01), 1)$
$= \delta(q_2, 1) = q_3$

$\hat{\delta}(q_0, 0111) = \delta(\hat{\delta}(q_0, 011), 1)$
$= \delta(q_3, 1)$
$= q_2$

$\hat{\delta}(q_0, 01110) = \delta(\hat{\delta}(q_0, 0111), 0)$
$= \delta(q_2, 0)$
$= q_3$

$\hat{\delta}(q_0, 011101) = \delta(\hat{\delta}(q_0, 01110), 1)$
$= \delta(q_3, 1)$
$= q_2 \in F$

$\Rightarrow$ w is accepted by the given DFA

$w = 011101$
$\hat{\delta}(q_0, w)$
$= \delta(\hat{\delta}(q_0, 01110), 1)$
$\downarrow$
$\delta(\hat{\delta}(q_0, 0111), 0)$
$\vdots$

$\Rightarrow$ accepts all strings of 0's and 1's of even length and begins with 01

The languages of DFA

- $A = (Q, \Sigma, \delta, q_0, F)$, a DFA
- $L(A) = \{ w \mid \hat{\delta}(q_0, w) \in F \}$

is the language of the DFA A

i.e. "the set of strings w that takes the start state q_0 to one of the accepting state."

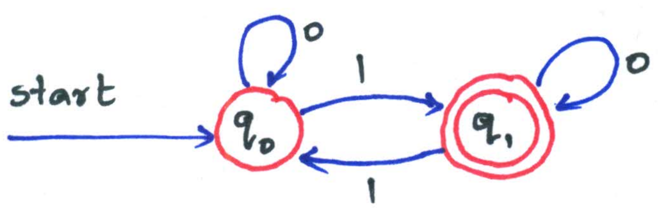
Regular Language / Regular Set

- L, a language over Σ is a regular language if $L = L(A)$ for some DFA A.

Example:

$L = \{ w \mid w \text{ is a binary string with odd numbers of 1's} \}$

is a regular language as the following DFA accepts L



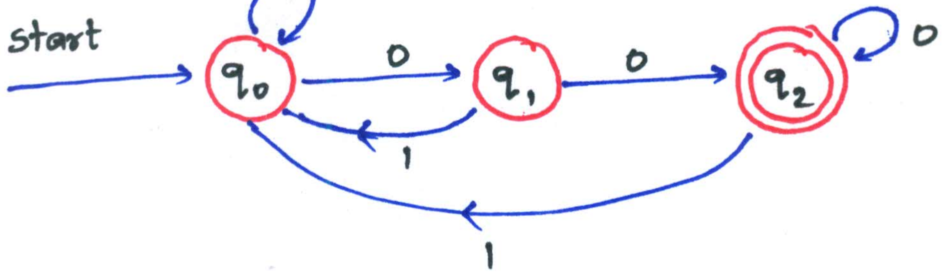
$1 \in L$

$111 \in L$

$01 \in L$

$1011 \in L$

Example:

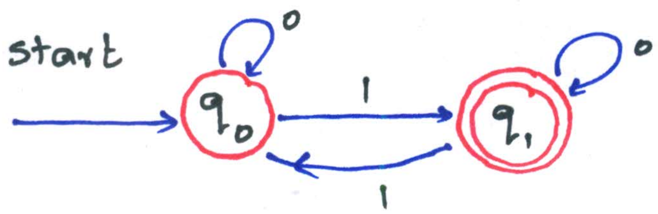


This DFA accepts all binary strings ending in 00

Example: Let us design a DFA to accept all strings of 0's and 1's with an odd number of 1's

L = {w | w is a binary string with odd number of 1}

Trial and error method [think / practice

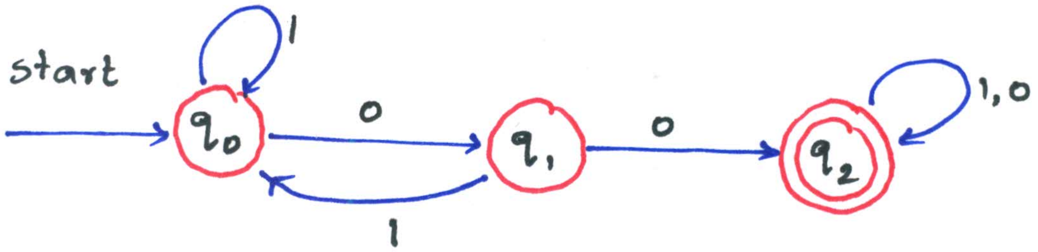


1
11 ×
01
000 . . . ×
010

Building DFA

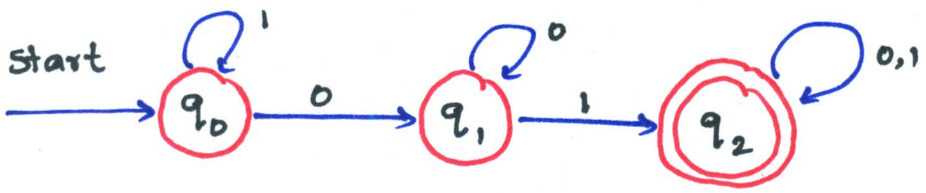
Example: Let L = {w | w is a binary string that contains the substring 00}

Design a DFA 'A' such that L(A) = L

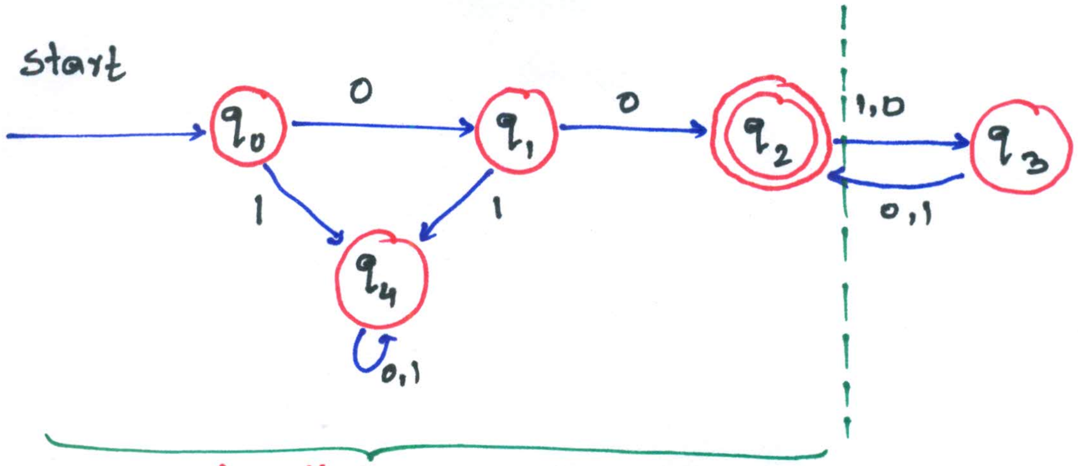


Example: Design a DFA that accepts the language

L = {w | w is a binary string that contains the substring 01}



Example: Build a DFA that accepts all binary strings starting with 00 and is of even length.



accepts all binary strings starting with 00

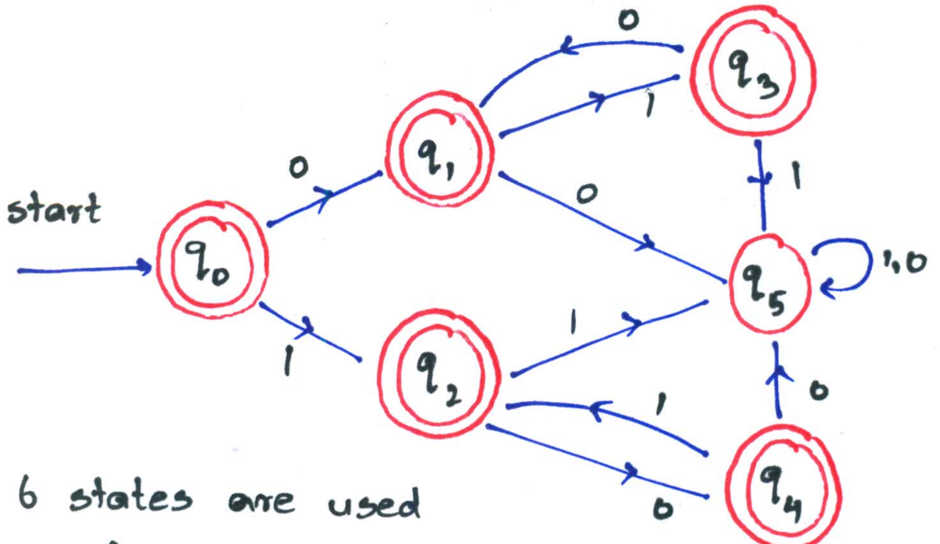
accepts all binary strings starting with 00 and having even length.

- $(Q, \Sigma, \delta, q_0, F)$

↳ F may have more than one state

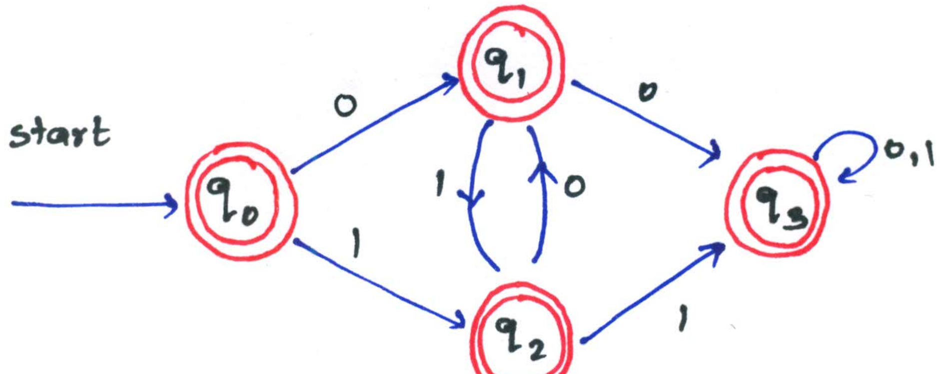
→ So, far we have considered only examples where F is a singleton set.

Example: A DFA that accepts all binary string where 0's and 1's are alternate



- 6 states are used
- # of accepting states - 5

• Can be done with fewer states.



- 4 states used
- # of accepting states : 3

Note:

- Given DFA corresponds to only one language
- Given language can have many DFA that accepts it
- DFA with minimum state is desirable
- Careful about the empty string (DFA should accept it or not)

Example: A DFA that accepts all binary strings that begin and end with same symbol